Artin's definition => Morphisms of sheaves

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ÉTALE COHOMOLOGY OF DIAMONDS

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ABSTRACT. Motivated by problems on the étale cohomology of Rapoport–Zink spaces and their generalizations, as well as Fargues's geometrization conjecture for the local Langlands correspondence, we develop a six functor formalism for the étale cohomology of diamonds, and more generally small v-stacks on the category of perfectoid spaces of characteristic p. Using a natural functor from analytic adic spaces over $\mathbf{Z}_p$ to diamonds which identifies étale sites, this induces a similar formalism in that setting, which in the noetherian setting recovers the formalism from Huber's book, [Hub96].

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1. Introduction

The aim of this manuscript is to lay foundations for the six operations in étale cohomology of adic spaces and diamonds, generalizing previous work of Huber, [Hub96].

In this manuscript, we will deal with analytic adic spaces X on which a fixed prime p is topologically nilpotent. Associated with any such X, we have an étale site $X_{\text{ét}}$ defined (under noetherian hypotheses) by Huber, [Hub96]. For any ring Λ such that $n\Lambda = 0$ for some n prime to p, we get a (left-completed) derived category $D_{\text{ét}}(X, \Lambda)$ of étale sheaves of Λ-modules on $X_{\text{ét}}$. It comes equipped with $- \otimes_\Lambda -$ and $R\mathscr{H}\mathrm{om}_\Lambda(-,-)$, and for a map $f : Y \to X$, one gets adjoint functors

$$f^* : D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda)$$

and

$$Rf_* : D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(X, \Lambda) .$$

Moreover, under certain conditions, one can define the pushforward with proper supports

$$Rf_! : D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(X, \Lambda) ,$$

with a right adjoint

$$Rf^! : D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda) .$$

If f is smooth, then (a form of) Poincaré duality identifies $Rf^!$ as a twist of f^*. Under suitable hypotheses, as for example for the adic spaces corresponding to rigid spaces over a fixed complete nonarchimedean base field, these results all appear in [Hub96].

Our aim here is to give a substantial generalization of this formalism. The motivation comes from some developments surrounding the theory of "local Shimura varieties" whose cohomology is expected to realize local Langlands correspondences for p-adic fields, and in particular Fargues' conjecture geometrizing the local Langlands correspondence through sheaves on the stack of G-bundles on the Fargues–Fontaine curve, [RV14, Far16, FS]. The formalism of this paper has recently been used by Kaletha–Weinstein for progress on the Kottwitz conjecture on the cohomology of local Shimura varieties, [KW17] (also assuming the forthcoming work [FS]).

In this context, it is necessary to work with certain objects that are not representable by adic spaces, but are merely quotients of adic spaces by "pro-étale" equivalence relations, akin to Artin's algebraic spaces. If one is willing to work with stacks, this brings objects like the classifying stack of $\mathrm{GL}_n(\mathbb{Q}_p)$ into the picture.

In fact, as any analytic adic space over $\mathbb{Z}_p$ is itself a quotient of a perfectoid space by a pro-étale equivalence relation, one can as well work with quotients of *perfectoid* spaces by pro-étale equivalence relations. By tilting, one can even assume that these perfectoid spaces are of characteristic p. This leads to the notion of a diamond. Let us quickly review the definition.

Definition 1.1. *A map $f : Y = \mathrm{Spa}(S, S^+) \to X = \mathrm{Spa}(R, R^+)$ of affinoid perfectoid spaces is affinoid pro-étale if it can be written as a cofiltered limit of étale maps $Y_i = \mathrm{Spa}(S_i, S_i^+) \to X$ of affinoid perfectoid spaces. More generally, a map $f : Y \to X$ of perfectoid spaces is pro-étale if it is locally on the source and target affinoid pro-étale.*

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Roland Huber

Étale Cohomology of Rigid Analytic Varieties and Adic Spaces

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A generalization of formal schemes and rigid analytic varieties

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0 Introduction

In this paper we construct a natural category $\mathscr{A}$ of locally and topologically ringed spaces which contains both the category of locally noetherian formal schemes and the category of rigid analytic varieties as full subcategories. This category has applications in algebraic geometry and rigid analytic geometry.

The idea of the definition of the category $\mathscr{A}$ is the following. From a formal point of view there is a certain similarity in constructing formal schemes and rigid analytic varieties. In both cases one starts with a certain class of topological rings (the adic rings in formal geometry and Tate algebras in rigid geometry), defines to every topological ring of this class a locally and topologically ringed space, and glueing of such spaces give formal schemes or rigid analytic varieties.

There is a natural class of topological rings which contains both the noetherian adic rings and the Tate algebras and which suggests itself. Namely the class of topological rings which have an open adic subring with a finitely generated ideal of definition. We call such a ring f-adic.

The points of the formal scheme $\mathrm{Spf}A$ associated with an adic ring A are the open prime ideals of A, and the points of the rigid analytic variety $\mathrm{Sp}A$ associated with a Tate algebra A are the maximal ideals of A. In both cases one can consider the points as continuous valuations of A. (A valuation $v\colon A \to \Gamma_v \cup \{0\}$ of a topological ring A is called continuous if the mapping v is continuous with respect to the ring topology of A and the order-induced topology of $\Gamma_v \cup \{0\}$.) Namely, if $\mathfrak{p}$ is an open prime ideal of an adic ring A then the trivial valuation $v_{\mathfrak{p}}$ of A with $v_{\mathfrak{p}}(a) = 0$ iff $a \in \mathfrak{p}$ is continuous, and if $\mathfrak{p}$ is a maximal ideal of a Tate algebra A over a valued field k then the

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valuation $v_\mathfrak{p}$ of A extending the valuation of k and with $v_\mathfrak{p}(a) = 0$ iff $a \in \mathfrak{p}$ is continuous.

This suggests to consider, for every f-adic ring A, the set $\mathrm{Spa}A$ of all continuous valuations of A. There is a natural topology on $\mathrm{Spa}A$. The topological space $\mathrm{Spa}A$ has been studied in [H1]. In this paper we will show that if A satisfies a certain condition of noetherianness then there is a natural sheaf $\mathcal{O}_A$ of topological rings on $\mathrm{Spa}A$. All stalks of $\mathcal{O}_A$ are local rings. Every continuous ring homomorphism $A \to B$ between f-adic rings induces a morphism of locally and topologically ringed spaces $(\mathrm{Spa}B, \mathcal{O}_B) \to (\mathrm{Spa}A, \mathcal{O}_A)$. We call a locally and topologically ringed space which is locally isomorphic to some $\mathrm{Spa}A = (\mathrm{Spa}A, \mathcal{O}_A)$ adic.(Here we suppress some details of the definition of adic spaces).

The functor $\mathrm{Spf}A \mapsto \mathrm{Spa}A$ from the category of noetherian affine formal schemes to the category of adic spaces extends in a natural way to a functor from the category of locally noetherian formal schemes to the category of adic spaces. Similarly, the functor $\mathrm{Sp}A \mapsto \mathrm{Spa}A$ from the category of affinoid rigid analytic varieties to the category of adic spaces extends to a functor from the category of rigid analytic varieties to the category of adic spaces. Both functors are fully faithful.

In the papers [H2] and [H3] we will study the coherent cohomology and the étale cohomology of adic spaces. Concerning the coherent cohomology, the main results are Theorem A and B, (i.e., every coherent $\mathcal{O}_A$-module $\mathcal{F}$ on an adic space $\mathrm{Spa}A$ is generated by its global sections and all higher cohomology groups $H^i(\mathrm{Spa}A, \mathcal{F})$ vanish) and the proper coherence theorem, (i.e., all direct image sheaves $R^n f_* \mathcal{F}$ of a coherent $\mathcal{O}_X$-module under a proper morphism of adic spaces $f: X \to Y$ are coherent $\mathcal{O}_Y$-modules). Many of the basic results of the étale cohomology of schemes also hold for the étale cohomology of adic spaces (for example, proper and smooth base change theorem, purity, Poincaré duality).

The theory of adic spaces may be useful for problems in algebraic geometry and rigid analytic geometry. For example, in [H3] we will show that the étale topos of a rigid analytic variety X is canonically equivalent to the étale topos of the adic space associated with X. Therefore all the results mentioned above on the étale cohomology of adic spaces also hold for the étale cohomology of rigid analytic varieties. In general, it is much easier to work with the étale toposes of rigid analytic varieties. For example, using geometric points of an adic space X, one can easily describe the category of points of the étale topos $X_{\acute{e}t}^{\sim}$ of X. (One obtains that $X_{\acute{e}t}^{\sim}$ has enough points.) But it is very complicated to describe within the category of rigid analytic varieties the points of the étale topos of a rigid analytic variety. Another application of the category of adic spaces will be given in Sect. 5 of this paper where we analyze Mumford's construction of semi-abelian group schemes within the category of adic spaces.

Let Perf denote the category of perfectoid spaces of characteristic p.[1] There are several different topologies one can consider on it, successively refining the previous.

(i) The analytic topology, where a cover $\{f_i : X_i \to X\}$ consists of open immersions $X_i \hookrightarrow X$ which jointly cover X.

(ii) The étale topology, where a cover $\{f_i : X_i \to X\}$ consists of étale maps $X_i \to X$ which jointly cover X.

(iii) The pro-étale topology, where a cover $\{f_i : X_i \to X\}$ consists of pro-étale maps $X_i \to X$ such that for any quasicompact open subset $U \subset X$, there are finitely many indices i and quasicompact open subsets $U_i \subset X_i$ such that the U_i jointly cover U.

(iv) The v-topology, where a cover $\{f_i : X_i \to X\}$ consists of any maps $X_i \to X$ such that for any quasicompact open subset $U \subset X$, there are finitely many indices i and quasicompact open subsets $U_i \subset X_i$ such that the U_i jointly cover U.

We note that the last part of the definition in cases (iii) and (iv) is the same condition that appears in the definition of the fpqc topology for schemes, and is automatic in cases (i) and (ii) as étale maps are open.

It turns out that all topologies are well-behaved; the v-topology is the finest, so for brevity we only state it in this case:

Theorem 1.2. *The v-topology on* Perf *is subcanonical, and for any affinoid perfectoid space* $X = \mathrm{Spa}(R, R^+)$, $H^0_v(X, \mathcal{O}_X) = R$, $H^0_v(X, \mathcal{O}_X^+) = R^+$ *and for* $i > 0$, $H^i_v(X, \mathcal{O}_X) = 0$ *and* $H^i_v(X, \mathcal{O}_X^+)$ *is almost zero.*

Analogous to Artin's theory of algebraic spaces, it is for the intended applications of the formalism necessary to allow quotients of perfectoid spaces by pro-étale equivalence relations. These are called diamonds:

Definition 1.3. *Let* Y *be a pro-étale sheaf on* Perf. *Then* Y *is a diamond if* Y *can be written as a quotient* $Y = X/R$ *of a perfectoid space* X *by a pro-étale equivalence relation* $R \subset X \times X$.

It turns out that all diamonds are v-sheaves. Now, for a diamond X, one can define sites X_v and $X_{\text{ét}}$. Actually, in the case of $X_{\text{ét}}$, one has to be careful, as in general there may not be enough étale maps into X. For this reason (and others), we often work with a restricted class of diamonds which turn out to be better-behaved. Here, we use the underlying topological space of a diamond Y, given as $|Y| = |X|/|R|$ in case $Y = X/R$ is a quotient of a perfectoid space X by a pro-étale equivalence relation R. There is a bijection between open subspaces of $|Y|$ and open subfunctors of Y, for an obvious definition of the latter. If Y is quasicompact as a v-sheaf, i.e. any collection of collectively surjective maps $Y_i \to Y$ of v-sheaves has a finite subcover, then $|Y|$ is quasicompact, but the converse is not true in general.

Definition 1.4. *A diamond* Y *is spatial if it is qcqs, and* $|Y|$ *admits a basis for the topology given by* $|U|$, *where* $U \subset Y$ *ranges over quasicompact open subdiamonds. More generally,* Y *is locally spatial if it admits an open cover by spatial diamonds.*

In particular, any perfectoid space X defines a locally spatial diamond, which is spatial precisely when X is qcqs. If X is a (locally) spatial diamond, then $|X|$ is a (locally) spectral topological

[1] For set-theoretic reasons, we (carefully) choose some cut-off cardinal κ, and work with κ-small perfectoid spaces throughout. The formalism does not depend on the choice of κ.

space, and X is quasicompact (resp. quasiseparated) as a v-sheaf precisely when $|X|$ is quasicompact (resp. quasiseparated) as a topological space. Now, if X is a locally spatial diamond, we define $X_{\text{ét}}$ as consisting of (locally separated) étale maps $Y \to X$ from diamonds Y (automatically locally spatial); it turns out that this is "big enough".

The following theorem shows that this generalizes the classical notions.

Theorem 1.5. *There is a natural functor from analytic adic spaces over $\mathbf{Z}_p$ to locally spatial diamonds, denoted $X \mapsto X^\diamond$, satisfying $X_{\text{ét}} \cong X^\diamond_{\text{ét}}$.*

Here, the functor $X \mapsto X^\diamond$ is intuitively given as follows. Choose a pro-étale surjection $\widetilde{X} \to X$ from a perfectoid space $\widetilde{X}$, and let $R \subset \widetilde{X} \times \widetilde{X}$ be the induced equivalence relation. This is pro-étale over $\widetilde{X}$, and thus perfectoid itself. Then

$$X^\diamond = \widetilde{X}^\flat / R^\flat \ .$$

Theorem 1.6. *Let X be a locally spatial diamond. Then pullback induces a fully faithful functor* [Hueb96] p. 19

$$\widehat{D}(X_{\text{ét}}, \Lambda) \to D(X_v, \Lambda) \ ,$$

where $\widehat{D}$ denotes the left-completion of the derived category; $D(X_v, \Lambda)$ is already left-complete.

Moreover, if $A \in D(X_v, \Lambda)$ and $f : Y \to X$ is a map of locally spatial diamonds that is surjective as a map of v-sheaves, then if f^*A lies in $\widehat{D}(Y_{\text{ét}}, \Lambda)$, then A lies in $\widehat{D}(X_{\text{ét}}, \Lambda)$.

Much of the formalism actually extends to all v-sheaves, or even v-stacks, subject to a small set-theoretic assumption of being "small".

Definition 1.7. *Let X be a small v-stack, and consider the site X_v of all perfectoid spaces over X, with the v-topology. Define the full subcategory*

$$D_{\text{ét}}(X, \Lambda) \subset D(X_v, \Lambda)$$

as consisting of all $A \in D(X_v, \Lambda)$ such that for all (equivalently, one surjective) map $f : Y \to X$ from a locally spatial diamond Y, f^*A lies in $\widehat{D}(Y_{\text{ét}}, \Lambda)$.

Now we can state that we have the following operations.

(i) A (derived) tensor product [Hueb96] p. 16

$$- \otimes^{\mathbb{L}}_{\Lambda} - : D_{\text{ét}}(X, \Lambda) \times D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X, \Lambda) \ .$$

This is compatible with the inclusions into $D(X_v, \Lambda)$, and the usual derived tensor product on X_v.

(ii) An internal Hom

$$R\mathscr{H}\mathrm{om}_\Lambda(-,-) : D_{\text{ét}}(X, \Lambda)^{\mathrm{op}} \times D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X, \Lambda)$$

characterized by the adjunction

$$R\,\mathrm{Hom}_{D_{\text{ét}}(X,\Lambda)}(A, R\mathscr{H}\mathrm{om}_\Lambda(B, C)) = R\,\mathrm{Hom}_{D_{\text{ét}}(X,\Lambda)}(A \otimes^{\mathbb{L}}_{\Lambda} B, C)$$

for all $A, B, C \in D_{\text{ét}}(X, \Lambda)$. In particular, for $A = \Lambda$,

$$R\Gamma(X, R\mathscr{H}\mathrm{om}_\Lambda(B, C)) = R\,\mathrm{Hom}_{D_{\text{ét}}(X,\Lambda)}(B, C) \ .$$

In general, the formation of $R\mathscr{H}\mathrm{om}_\Lambda$ does not commute with the inclusion $D_{\text{ét}}(X, \Lambda) \subset D(X_v, \Lambda)$.

(iii) For any map $f : Y \to X$ of small v-stacks, a pullback functor

$$f^* : D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda) .$$

This is compatible with the inclusions into $D(X_v, \Lambda)$ resp. $D(Y_v, \Lambda)$, and the pullback functor $D(X_v, \Lambda) \to D(Y_v, \Lambda)$.

(iv) For any map $f : Y \to X$ of small v-stacks, a pushforward functor

$$Rf_* : D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(X, \Lambda)$$

which is right adjoint to f^*. In general, formation of Rf_* does not commute with the inclusions into $D(X_v, \Lambda)$ resp. $D(Y_v, \Lambda)$, but this holds true if f is qcqs and one starts with an object of D^+.

(v) For any map $f : Y \to X$ of small v-stacks that is compactifiable (cf. Definition 22.2), representable in locally spatial diamonds (cf. Definition 13.3) and with $\dim.\,\mathrm{trg}\, f < \infty$ (cf. Definition 21.7), a functor

$$Rf_! : D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(X, \Lambda) .$$

(vi) For any map $f : Y \to X$ of small v-stacks that is compactifiable, representable in locally spatial diamonds and with $\dim.\,\mathrm{trg}\, f < \infty$, a functor

$$Rf^! : D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda)$$

that is right adjoint to $Rf_!$.

The first four operations are defined in Section 17, and then $Rf_!$ is defined in Section 22, in particular Definition 22.18, and $Rf^!$ is defined in Section 23.

These operations satisfy the following formulas.

Theorem 1.8. *Let $f : Y \to X$ be a map of small v-stacks.*

(i) *For all $A, B \in D_{\text{ét}}(X, \Lambda)$, one has*

$$f^*A \otimes^{\mathbb{L}}_{\Lambda} f^*B \cong f^*(A \otimes^{\mathbb{L}}_{\Lambda} B) .$$

(ii) *For all $A \in D_{\text{ét}}(X, \Lambda)$, $B \in D_{\text{ét}}(Y, \Lambda)$, one has*

$$Rf_* R\mathscr{H}\mathrm{om}_{\Lambda}(f^*A, B) \cong R\mathscr{H}\mathrm{om}_{\Lambda}(A, Rf_*B) .$$

(iii) *Assume that f is compactifiable and representable in locally spatial diamonds with* $\dim.\,\mathrm{trg}\, f < \infty$. *For all $A \in D_{\text{ét}}(X, \Lambda)$, $B \in D_{\text{ét}}(Y, \Lambda)$, one has*

$$Rf_!(A \otimes^{\mathbb{L}}_{\Lambda} f^*B) \cong Rf_!A \otimes^{\mathbb{L}}_{\Lambda} B .$$

(iv) *Assume that f is compactifiable and representable in locally spatial diamonds with* $\dim.\,\mathrm{trg}\, f < \infty$. *For all $A \in D_{\text{ét}}(Y, \Lambda)$, $B \in D_{\text{ét}}(X, \Lambda)$, one has*

$$R\mathscr{H}\mathrm{om}_{\Lambda}(Rf_!A, B) \cong Rf_*R\mathscr{H}\mathrm{om}_{\Lambda}(A, Rf^!B) .$$

(v) *Assume that f is compactifiable and representable in locally spatial diamonds with* $\dim.\,\mathrm{trg}\, f < \infty$. *For all $A, B \in D_{\text{ét}}(X, \Lambda)$, one has*

$$Rf^!R\mathscr{H}\mathrm{om}_{\Lambda}(A, B) \cong R\mathscr{H}\mathrm{om}_{\Lambda}(f^*A, Rf^!B) .$$

Parts (i) and (ii) are proved in Section 17. Part (iii) is Proposition 22.23, and parts (iv) and (v) are formal consequences of (iii) and adjunctions, cf. Proposition 23.3.

Moreover, there are the following base change results.

Theorem 1.9. *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{\tilde{g}} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

be a cartesian diagram of small v-stacks.

(i) *Assume that f is qcqs, and $A \in D^+_{\text{ét}}(Y, \Lambda)$. Then*

$$g^* Rf_* A \cong Rf'_* \tilde{g}^* A .$$

(ii) *Assume that f is compactifiable and representable in locally spatial diamonds with* $\dim.\operatorname{trg} f < \infty$, *and $A \in D_{\text{ét}}(Y, \Lambda)$. Then*

$$g^* Rf_! A \cong Rf'_! \tilde{g}^* A .$$

(iii) *Assume that g is compactifiable and representable in locally spatial diamonds with* $\dim.\operatorname{trg} f < \infty$, *and $A \in D_{\text{ét}}(Y, \Lambda)$. Then*

$$Rg^! Rf_* A \cong Rf'_* R\tilde{g}^! A .$$

Part (i) is Proposition 17.6, part (ii) is Proposition 22.19, and part (iii) is Proposition 23.16.

We also need a theory of smooth morphisms. From now on, fix a prime $\ell \neq p$, and assume that Λ is killed by a power of ℓ. In Definition 23.8, we define a notion of ℓ-cohomologically smooth morphisms for any prime $\ell \neq p$ (which depends on the prime ℓ), which includes étale maps.

Theorem 1.10. *Let $f : Y \to X$ be a map of small v-stacks that is separated and representable in locally spatial diamonds with* $\dim.\operatorname{trg} f < \infty$. *The property of f being ℓ-cohomologically smooth is v-local on the target and ℓ-cohomologically smooth local on the source; moreover, composites and base changes of ℓ-cohomologically smooth maps are ℓ-cohomologically smooth, and smooth maps of analytic adic spaces over $\mathbf{Z}_p$ give rise to ℓ-cohomologically smooth morphisms under $X \mapsto X^\diamond$. Assume that f is ℓ-cohomologically smooth, and Λ is ℓ-power torsion.*

(i) *There is a natural equivalence of functors*

$$Rf^! A \cong f^* A \otimes^{\mathbb{L}}_{\Lambda} Rf^! \Lambda : D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda) ,$$

where $Rf^! \Lambda$ is v-locally isomorphic to $\Lambda[n]$ for some $n \in \mathbb{Z}$.

(ii) *For $A, B \in D_{\text{ét}}(X, \Lambda)$, there is a natural equivalence*

$$f^* R\mathscr{H}\mathrm{om}_\Lambda(A, B) \cong R\mathscr{H}\mathrm{om}_\Lambda(f^* A, f^* B) .$$

(iii) *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{\tilde{g}} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

be a cartesian diagram of small v-stacks, with f being ℓ-cohomologically smooth as before. Then for all $A \in D_{\text{ét}}(X', \Lambda)$,

$$f^* Rg_* A \cong R\tilde{g}_* f'^* A .$$

Moreover, for all $A \in D_{\text{ét}}(X, \Lambda)$,

$$\tilde{g}^* Rf^! A \cong Rf'^! g^* A$$

[Hub96] p. 8

and

$$f'^* Rg^! A \cong R\tilde{g}^! f^* A \ .$$

We refer to Sections 23 and 24 for these results.

Moreover, for ℓ-cohomologically smooth morphisms, one can prove some finiteness results. For this, we restrict to the case $\Lambda = \mathbb{F}_\ell$. For any small v-stack X, one can define a full subcategory $D_{\text{cons}}(X, \mathbb{F}_\ell) \subset D_{\text{ét}}(X, \mathbb{F}_\ell)$; containment can be checked v-locally, and on affinoid perfectoid spaces, those are bounded complexes that become locally constant and finite-dimensional in each degree over a *constructible* stratification. Note that constructible here refers to the standard notion for spectral spaces, and excludes the stratification of a closed unit disc into a point and its complement (as the open complement is not quasicompact). We note that this condition can be checked on cohomology sheaves. The following theorem is also proved in Section 23.

Theorem 1.11. *Let $f : Y \to X$ be a quasicompact ℓ-cohomologically smooth map (in particular, compactifiable, representable in locally spatial diamonds, and with* $\dim.\,\mathrm{trg}\, f < \infty$*) of small v-stacks. For all $A \in D_{\text{cons}}(Y, \mathbb{F}_\ell)$, one has $Rf_! A \in D_{\text{cons}}(X, \mathbb{F}_\ell)$.*

[Hub96]

Another finiteness result is the following, cf. Theorem 25.1.

Def. 1.2.5 p. 50

Theorem 1.12. *Let C be a complete algebraically closed nonarchimedean field of characteristic p with ring of integers $\mathcal{O}_C$, and let X be a locally spatial diamond that is separated and ℓ-cohomologically smooth over $\mathrm{Spa}(C, \mathcal{O}_C)$ for some $\ell \neq p$. Let $A \in D_{\text{ét}}(X, \mathbb{F}_\ell)$ be a bounded complex with constructible cohomology. Then the double (naive) duality map*

[Hub96] p. 23

$$A \to R\mathscr{H}\mathrm{om}_{\mathbb{F}_\ell}(R\mathscr{H}\mathrm{om}_{\mathbb{F}_\ell}(A, \mathbb{F}_\ell), \mathbb{F}_\ell)$$

is an equivalence; equivalently, as $Rf^! \mathbb{F}_\ell$ is invertible, the double Verdier duality map

[Hub96] p. 23

$$A \to R\mathscr{H}\mathrm{om}_{\mathbb{F}_\ell}(R\mathscr{H}\mathrm{om}_{\mathbb{F}_\ell}(A, Rf^! \mathbb{F}_\ell), Rf^! \mathbb{F}_\ell)$$

is an equivalence. Moreover, if X is quasicompact, then $H^i(X, A)$ is finite for all $i \in \mathbb{Z}$.

Finally, it is useful to have some "invariance under change of algebraically closed base field" statements. These come in different flavours, depending on whether the base field is discrete or nonarchimedean. The following is Theorem 19.5.

Theorem 1.13. *Let X be a small v-stack.*

(i) *Assume that X lives over k, where k is a discrete algebraically closed field of characteristic p, and k'/k is an extension of discrete algebraically closed base fields, $X' = X \times_k k'$. Then the pullback functor*

[Hub96] p. 19

$$D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X', \Lambda)$$

is fully faithful.

(ii) *Assume that X lives over k, where k is an algebraically closed discrete field of characteristic p. Let C/k be an algebraically closed complete nonarchimedean field, and $X' = X \times_k \mathrm{Spa}(C, C^+)$ for some open and bounded valuation subring $C^+ \subset C$ containing k. Then the pullback functor*

[Hub96] p. 19

$$D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X', \Lambda)$$

is fully faithful.

(iii) *Assume that X lives over $\mathrm{Spa}(C, C^+)$, where C is an algebraically closed complete nonarchimedean field with an open and bounded valuation subring $C^+ \subset C$, C'/C is an extension of algebraically closed complete nonarchimedean fields, and $C'^+ \subset C'$ an open and bounded valuation subring containing C^+, such that $\mathrm{Spa}(C', C'^+) \to \mathrm{Spa}(C, C^+)$ is surjective. Then for $X' = X \times_{\mathrm{Spa}(C,C^+)} \mathrm{Spa}(C', C'^+)$, the pullback functor*

$$D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X', \Lambda)$$

is fully faithful.

Let us say some words about the proofs. We start with the proof of Theorem 1.2. The proof proceeds in steps, proving first the analytic and étale cases (already in [Sch12], [KL15]), deducing the pro-étale case by a limit argument, and then the v-case by making use of the notion of (strictly) totally disconnected spaces.

Definition 1.14. *Let X be a perfectoid space. Then X is totally disconnected (resp. strictly totally disconnected) if X is quasicompact, and every open cover of X splits (resp. and every étale cover of X splits).*

It turns out that totally disconnected spaces have a very simple form.

Proposition 1.15. *Let X be a perfectoid space. Then X is totally disconnected (resp. strictly totally disconnected) if and only if X is affinoid, and every connected component of X is of the form $\mathrm{Spa}(K, K^+)$, where K is a perfectoid field with an open and bounded valuation subring $K^+ \subset K$ (resp. and K is algebraically closed).*

Thus, (strictly) totally disconnected are essentially profinite sets of (geometric) points. Their use is justified by the following two lemmas.

Lemma 1.16. *Let X be any quasicompact perfectoid space. Then there is a pro-étale cover $\widetilde{X} \to X$ such that $\widetilde{X}$ is strictly totally disconnected.*

Lemma 1.17. *Let $X = \mathrm{Spa}(R, R^+)$ be a totally disconnected perfectoid space, and $f : Y = \mathrm{Spa}(S, S^+) \to X$ be any map of perfectoid spaces. Then $R^+/\varpi \to S^+/\varpi$ is flat, for any pseudouniformizer $\varpi \in R$.*

Using the first lemma and pro-étale descent, general v-descent reduces to the case of strictly totally disconnected spaces. Then the second lemma reduces this to faithfully flat descent.

Another important use of strictly totally disconnected spaces is a classification result for pro-étale maps. This uses the notion of separated morphisms of perfectoid spaces, for which we refer to Definition 5.10.

Lemma 1.18. *Let X be a strictly totally disconnected perfectoid spaces. Then a quasicompact separated map $f : Y \to X$ of perfectoid spaces is pro-étale if and only if for every rank-1-point $x = \mathrm{Spa}(C(x), \mathcal{O}_{C(x)}) \in X$, the fibre of f over x is isomorphic to $\underline{S_x} \times \mathrm{Spa}(C(x), \mathcal{O}_{C(x)})$ for some profinite set S_x. In this case, f is affinoid pro-étale.*

Many arguments follow this pattern of reduction to (strictly) totally disconnected spaces, which are essentially just profinite sets. This is akin to covering compact Hausdorff spaces by profinite sets. Making these arguments work is often a question of delicate point-set topology, and we will need to use the properties of spectral spaces very intensely. One application of this is v-descent for separated étale maps:

Huber (p.2) open resp. locally closed constructible subsets

Info is German impression; all remaining sentences are (have been) quoted from Huber-book !!!

[Hub96]
(1.1.13) p.44

2. Spectral Spaces

We recall some basic properties of spectral spaces that are used throughout.

[Hub96] L.0.2.2; p.3 p.1, §0.1 p.2, §0.1 p.1, §0.1 p.2, §0.1

Definition 2.1. *A spectral space is a topological space X such that X is quasicompact, X has a basis of quasicompact open subsets stable under finite intersection, and every irreducible closed subset has a unique generic point. A locally spectral space is a topological space X that admits an open cover by spectral subspaces.*

Every locally closed constructible subset $S \subset X$ [Hub96] p.2, §0.1 p.4.

A morphism $f : X \to Y$ between spectral spaces is spectral if it is continuous and the preimage of any quasicompact open subset is quasicompact open. A morphism $f : X \to Y$ between locally spectral spaces is spectral if it is continuous and for any spectral open subspace $U \subset X$ mapping into some spectral open $V \subset Y$, the map $U \to V$ is spectral.

[Hub96] p.44

A subset S of X is called: open constructible

Theorem 2.2 (Hochster)**.** *Let X be a topological space. The following conditions are equivalent.*

(i) *The space X is spectral.*

(ii) *There is a ring A such that $X \cong \operatorname{Spec} A$.*

(iii) *The space X can be written as an inverse limit of finite T_0-spaces.*

The category of spectral spaces with spectral maps is equivalent to the pro-category of finite T_0-spaces.

[Hub96] p.1, §0.1

Recall that a subset $T \subset X$ of a spectral space X is constructible if it lies in the boolean algebra generated by quasicompact open subsets. The constructible topology on X is the topology generated by constructible subsets. Then X equipped with its constructible topology is a profinite set. If X is an inverse limit of finite T_0-spaces X_i, then X with the constructible topology is the inverse limit of the X_i equipped with the discrete topology. [Hub96] (1.1.13), p.44

[Hub96] p.2, §0.1 locally closed constructible subset

A subset $T \subset X$ is pro-constructible if it is an intersection of constructible subsets. More generally, a subset $T \subset X$ of a locally spectral space X is constructible (resp. pro-constructible) if the intersection with any spectral open subspace is constructible (resp. pro-constructible). [Hub96] (1.1.13), p.44

Lemma 2.3. *Let X be a spectral space. A subset $S \subset X$ is pro-constructible if and only if S is closed in the constructible topology. If $f : X \to Y$ is a spectral map of spectral spaces, then the image of any pro-constructible subset of X is a pro-constructible subset of Y.*

[Hub96] p.2 p.3

Proof. Constructible subsets are open and closed in the constructible topology, so pro-constructible subsets are closed. Conversely, if a subset is closed in the constructible topology, then it is an intersection of open and closed subsets in the constructible topology. But open and closed subsets of the constructible topology are constructible, for example by writing X as a cofiltered inverse limit of finite T_0-space X_i (where any open and closed subset of X for the constructible topology is a preimage of a subset of some X_i). [Hub96] (1.1.13), p.44

[Hub96] p.2; §0.1

Now let $f : X \to Y$ be spectral map of spectral spaces, and $S \subset X$ a pro-constructible subset with image $T \subset Y$. If we equip X and Y with their constructible topology, then f is a continuous map of compact Hausdorff spaces, and $S \subset X$ is a closed subset, which thus has closed image $T \subset Y$. Thus, T is pro-constructible. □ [Hub96] (1.1.13), p.44

[Hub96] L.0.2.2; p.3

Lemma 2.4. *Let X be a spectral space, and $S \subset X$ a pro-constructible subset. Then the closure $\bar{S} \subset X$ of S is the set of specializations of points in S.*

{t}

[Hub.96] Prop. 1.3.8, p.53 spectral space (Scholze) = special subspaces (Huber) [Hub96] p.4!

Theorem 1.19. *The following prestacks are v-stacks.*

(i) *The prestack of affinoid perfectoid spaces over the category of totally disconnected perfectoid spaces.*

(ii) *The prestack of separated pro-étale perfectoid spaces over the category of strictly totally disconnected perfectoid spaces.*

(iii) *The prestack of separated étale maps over the category of all perfectoid spaces.*

Using these descent theorems, the following definition becomes reasonable.

Definition 1.20. *A locally separated map $f : Y \to X$ of v-stacks is étale (resp. quasi-pro-étale) if for any perfectoid space (resp. any strictly totally disconnected perfectoid space) Z mapping to X, the fibre product $Y \times_X Z$ is representable by a perfectoid space, and étale (resp. pro-étale) over Z.*

This leads to the following characterization of diamonds.

Proposition 1.21. *A v-sheaf Y is a diamond if and only if there is a surjective quasi-pro-étale map $X \to Y$ from a perfectoid space X.*

A positive side of the v-topology is that any small v-stack can be accessed through the following steps: from perfectoid spaces to diamonds by quotients under a pro-étale equivalence relation; from diamonds to general small v-sheaves by quotients under a diamond equivalence relation; and from small v-sheaves to small v-stacks by quotients under a small v-sheaf equivalence relation. In the key step, one uses that any sub-v-sheaf of a diamond is automatically itself a diamond; this statement does not seem to have any classical analogue.

However, at some point, we have to prove some theorems, in particular invariance under change of algebraically closed base field, smooth and proper base change, and Poincaré duality. Invariance under change of algebraically closed base field is reduced by approximation to the noetherian cases handled by Huber (which in turn are reduced to the case of schemes via nearby cycles). Proper base change is reduced through a series of reductions to a statement about Zariski–Riemann spaces of (algebraically closed) fields, which follows from proper base change for schemes. For smooth base change and Poincaré duality, things are reduced by a series of reductions to the case of a closed unit disc, where we can use Huber's results. We note that our reductions reduce all appeals to Huber's book to the case of smooth curves over $\mathrm{Spa}(C, C^+)$, where C is an algebraically closed complete nonarchimedean field, and $C^+ \subset C$ is an open and bounded valuation subring; however, in that case, we need a solid theory of Poincaré duality.

We make a general warning to the reader that this manuscript contains essentially no examples, which probably makes it almost unreadable. We hope that forthcoming papers, including [FS] and the recent paper of Kaletha–Weinstein, [KW17], will give the reader the necessary examples and motivation to work through the long technical arguments of this paper.

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0 Étale cohomology of rigid analytic varieties (summary)

In this chapter we summarize our results on the étale cohomology of rigid analytic varieties which will be proved more generally for adic spaces in later chapters. Our general reference for rigid analytic varieties is [BGR], [FP].

0.1 The étale site of a rigid analytic variety

Let k be a complete non-archimedean field. Put $\mathbb{B}^n := \mathrm{Sp}(k\langle T_1, \ldots, T_n\rangle)$.

Let $f: X \to Y$ be a morphism of rigid analytic varieties over k. f is called étale if, for every $x \in X$ and $y := f(x) \in Y$, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_{Y,y}\mathcal{O}_{X,x}$ is a finite separable extension field of $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}$. f is called smooth if every $x \in X$ has an admissible open neighbourhood U such that the restriction $f|U: U \to Y$ has a factorization $U \xrightarrow{g} Y \times \mathbb{B}^n \xrightarrow{h} Y$ where g is étale and h is the projection.

For a rigid analytic variety X, let $\acute{\mathrm{E}}\mathrm{t}/X$ denote the category of rigid analytic varieties étale over X. The étale site $X_{\acute{e}t}$ of X is the category $\acute{\mathrm{E}}\mathrm{t}/X$ equipped with the topology such that a family $(Y_i \xrightarrow{\varphi_i} Y)_{i\in I}$ of morphisms in $\acute{\mathrm{E}}\mathrm{t}/X$ is a covering if and only if, for every quasi-compact admissible open subset V of Y, there are quasi-compact admissible open subsets $U_1, \ldots, U_n$ of $\coprod_{i\in I} Y_i$ such that $V = \varphi(\bigcup_{j=1}^{n} U_j)$ where $\varphi: \coprod_{i\in I} Y_i \to Y$ is the morphism induced by the φ_i ([SS]).

Example 0.1.1. Let X be a rigid analytic variety over k.

i) Putting $\mathbb{G}_m(Y) := \mathcal{O}_Y(Y)^*$ for every $Y \in \mathrm{ob}\acute{\mathrm{E}}\mathrm{t}/X$, we obtain a sheaf $\mathbb{G}_m$ on $X_{\acute{e}t}$. The natural mapping $H^1(X, \mathcal{O}_X^*) \to H^1(X_{\acute{e}t}, \mathbb{G}_m)$ is bijective.

ii) For every abelian torsion sheaf F on $X_{\acute{e}t}$ whose torsion is prime to $\mathrm{char}(k^\circ/k^{\circ\circ})$ and for every $n \in \mathbb{N}_0$, the natural mapping $H^n(X, F) \to H^n(X \times \mathbb{B}^n, t^*F)$ is bijective where $t: X \times \mathbb{B}^n \to X$ is the projection. One of the basic ingredients of the proof of this statement is explained in the next section.

0.2 Cohomology of constant sheaves

(For the notions of open subtoposes, complementary closed subtoposes ... we refer to [SGA 4].)

Let $X = \mathrm{Sp}A$ be an affinoid rigid analytic variety over a complete non-archimedean field k. We call a subtopos S of the étale topos $X_{\text{ét}}^{\sim}$

- *open constructible* if S is coherent and an open subtopos of $X_{\text{ét}}^{\sim}$
- *closed constructible* if S is a closed subtopos of $X_{\text{ét}}^{\sim}$ complementary to an open constructible subtopos of $X_{\text{ét}}^{\sim}$
- *locally closed constructible* if S is the intersection of an open constructible subtopos of $X_{\text{ét}}^{\sim}$ and a closed constructible subtopos of $X_{\text{ét}}^{\sim}$.

p. 10 Def. 2.1

The subtoposes of $X_{\text{ét}}^{\sim}$ correspond bijectively to the topologies of the category $\text{Ét}/X$ which are stronger than the topology of $X_{\text{ét}}$ ([SGA 4, IV.9.1.2.c]). Let me describe the topologies of $\text{Ét}/X$ which correspond to the locally closed constructible subtoposes of $X_{\text{ét}}^{\sim}$.
For that we define: A subset S of X is called

- *open constructible* if S is a quasi-compact admissible open subset of X
- *closed constructible* if $S = X - U$ is the complement of an open constructible subset of X
- *locally closed constructible* if S is the intersection of an open constructible subset of X and a closed constructible subset of X.

Every locally closed constructible subset S of X is an admissible open subset of X. Hence we have the étale site $S_{\text{ét}}$ of the rigid analytic variety given by S. But we can associate with S also the following site $S_{\text{ét},X}$: The category underlying $S_{\text{ét},X}$ is the category $\text{Ét}/X$. A family $(Y_i \xrightarrow{\varphi_i} Y)_{i\in I}$ of morphisms in $\text{Ét}/X$ is a covering of Y in $S_{\text{ét},X}$ if and only if, for every quasi-compact admissible open subset V of Y, there are quasi-compact admissible open subsets $U_1, \ldots, U_n$ of $\coprod_{i\in I} Y_i$ such that $g^{-1}(S) \cap V \subseteq \varphi(\bigcup_{j=1}^{n} U_j) \subseteq V$ where $g: Y \to X$ is the structure morphism and $\varphi: \coprod_{i\in I} Y_i \to Y$ is the morphism induced by the φ_i.

Let $S_{\text{ét},X}^{\sim}$ denote the topos of the site $S_{\text{ét},X}$. The sites $X_{\text{ét}}$ and $S_{\text{ét},X}$ have the same underlying category, and the topology of $S_{\text{ét},X}$ is stronger than the topology of $X_{\text{ét}}$. Hence $S_{\text{ét},X}^{\sim}$ is a subtopos of $X_{\text{ét}}^{\sim}$. The mapping $S \mapsto S_{\text{ét},X}^{\sim}$ is a bijection from the set of locally closed (resp. closed, resp. open) constructible subsets of X onto the set of locally closed (resp. closed, resp. open) constructible subtoposes of $X_{\text{ét}}^{\sim}$.

4.2.3, p. 10

As already mentioned above, every locally closed constructible subset S of X is an admissible open subset of X, and so we have the site $S_{\text{ét}}$. There is a natural morphism of sites

$$j: S_{\acute{e}t} \longrightarrow S_{\acute{e}t,X}.$$

The induced morphism of toposes

$$S^{\sim}_{\acute{e}t} \longrightarrow S^{\sim}_{\acute{e}t,X}$$

is an equivalence if and only if S is an open constructible subset of X. (It is not unlikely that, for every locally closed constructible subset S of X and constant ℓ-torsion sheaf G of finite type on $S_{\acute{e}t,X}$ with ℓ invertible in the valuation ring of the non-archimedean field k, the mapping $H^n(S_{\acute{e}t,X}, G) \to H^n(S_{\acute{e}t}, j^*G)$ $(n \in \mathbb{N}_0)$ is bijective.)

We want to get some informations on the cohomology groups $H^n(S_{\acute{e}t,X}, G)$ where S is a locally closed constructible subset of X and G is the constant sheaf on $S_{\acute{e}t,X}$ to an abelian torsion group. For that we first introduce a special class of locally closed constructible subsets of X.

Definition 0.2.1. A subset S of $X = \mathrm{Sp}A$ is called *special* if there exist finite subsets D and E of A and an element t of A such that $A = (D \cup E \cup \{t\}) \cdot A$ and

$$S = \{x \in X \mid |d(x)| \leq |t(x)| \quad \text{for every} \quad d \in D \\ \text{and } |e(x)| < |t(x)| \quad \text{for every} \quad e \in E\}.$$

A subset S of X is called *closed special* if there exist a finite subset E of A and an element t of A such that $A = (E \cup \{t\}) \cdot A$ and

$$S = \{x \in X \mid |e(x)| < |t(x)| \quad \text{for every} \quad e \in E\}.$$

Every rational subset of X is special. We have

Def. 2.1, p. 10
Lem. 2.4, p. 10

Lemma 0.2.2. i) *Let S be a special subset of X. Then S is a locally closed constructible subset of X. $S^{\sim}_{\acute{e}t,X}$ is an open subtopos of $X^{\sim}_{\acute{e}t}$ if and only if S is a rational subset of X, and $S^{\sim}_{\acute{e}t,X}$ is a closed subtopos of $X^{\sim}_{\acute{e}t}$ if and only if S is a closed special subset of X.*

ii) *The intersection of two special subsets of X is special. Every locally closed constructible subset S of X is a finite union of special subsets, $S = \bigcup_{i=1}^{n} S_i$, such that, for every $i \in \{1, \ldots, n\}$, $(S_i)^{\sim}_{\acute{e}t,X}$ is an open subtopos of $S^{\sim}_{\acute{e}t,X}$.*

iii) *The intersection of two closed special subsets of X is closed special. Every closed constructible subset S of X is a finite union $S = \bigcup_{i=1}^{n} S_i$ of closed special subsets of X.*

Let S be a locally closed constructible subset of X. According to Lemma (0.2.2), in order to compute $H^n(S_{\acute{e}t,X}, G)$ we can apply the spectral sequence for open coverings (or closed coverings if S is closed constructible) and thus reduce to the case that S is a special subset of X. For special subsets we have the following theorem.

Theorem 0.2.3. *Let S be a special subset of $X = \mathrm{Sp}A$. Let G be a set (resp. ind-finite group, resp. abelian torsion group). Then, for $n = 0$ (resp. $n \in \{0, 1\}$, resp. $n \in \mathbb{N}_0$), there is a natural isomorphism*

$$H^n(S_{\acute{e}t,X}, G) \cong H^n((\mathrm{Spec}A(S))_{\acute{e}t}, G)$$

where the ring $A(S)$ is defined as follows: Let

$$\begin{aligned} S \;=\; \{x \in X \mid & |d(x)| \leq |t(x)| \quad \text{for every} \quad d \in D \quad \text{and} \\ & |e(x)| < |t(x)| \quad \text{for every} \quad e \in E\} \end{aligned}$$

be a representation of S as in Definition (0.2.1). Let B be the henselization of the ring $A^\circ[\frac{f}{t} \mid f \in D \cup E] \subseteq A_t$ along the ideal generated by $A^{\circ\circ} \cup \{\frac{e}{t} \mid e \in E\}$. Then

$$A(S) := B \otimes_{k^\circ} k.$$

The ring $A(S)$ depends only on S. (A° denotes the ring of power-bounded elements of A, and $A^{\circ\circ}$ denotes the ideal of topologically nilpotent elements of A°.)

In (0.2.3), instead of the ring A° one can use other subrings of A. We formulate this in the following stronger version of Theorem (0.2.3).

Theorem 0.2.4. *Let k be the non-archimedean field over which $X = \mathrm{Sp}A$ is defined. Let A_0 be a sub-k°-algebra of A such that the subspace topology of A on A_0 is the sA_0-adic topology for some $s \in k^{\circ\circ} - \{0\}$ and the completion $\hat{A}_0$ of A_0 is a lattice of A, i.e. A_0 is open in A and of topologically finite type over k°. For every special subset S of X,*

$$\begin{aligned} S \;=\; \{x \in X \mid & |d(x)| \leq |t(x)| \quad \text{for every} \quad d \in D \quad \text{and} \\ & |e(x)| < |t(x)| \quad \text{for every} \quad e \in E\} \end{aligned}$$

(where D, E are finite subsets of A and t is an element of A with $A = (D \cup E \cup \{t\}) \cdot A$), put

$$A_0(S) := B \otimes_{k^\circ} k$$

where B denotes the henselization of the ring $A_0[\frac{f}{t} \mid f \in D \cup E] \subseteq A_t$ along the ideal generated by $k^{\circ\circ} \cup \{\frac{e}{t} \mid e \in E\}$. The ring $A_0(S)$ depends only on A_0 and S (and is independent of the choice of D, E, t).
Let G be a set (resp. ind-finite group, resp. abelian torsion group). Then, for $n = 0$ (resp. $n \in \{0,1\}$, resp. $n \in \mathbb{N}_0$), there is a natural isomorphism

$$H^n(S_{\text{ét},X}, G) \cong H^n((\operatorname{Spec} A_0(S))_{\text{ét}}, G).$$

Theorem (0.2.4) (and its generalization to adic spaces) is fundamental for this book. It can be applied to compute concrete étale cohomology groups of rigid analytic varieties (cf. Example (0.2.5) below), but it is also the basic ingredient for the proofs of many general statements on the étale cohomology of rigid analytic varieties (cf. (0.5.2)).

Example 0.2.5. As an application of Theorem (0.2.4) let us compute the following cohomology group.
Let k be an algebraically closed complete non-archimedean field. Let $(\mathbb{P}^1_k)^{\text{rig}} = k \cup \{\infty\}$ be the rigid analytic variety associated with the projective line $\mathbb{P}^1_k$. Then

$$U := \{x \in (\mathbb{P}^1_k)^{\text{rig}} \mid |x| \leq 1\}$$

is an admissible open subset of $(\mathbb{P}^1_k)^{\text{rig}}$. Let

$$j : U \hookrightarrow (\mathbb{P}^1_k)^{\text{rig}}$$

be the open embedding. Let G be a $\mathbb{Z}/m\mathbb{Z}$-module with m prime to $\operatorname{char}(k^\circ/k^{\circ\circ})$. Then

$$H^n((\mathbb{P}^1_k)^{\text{rig}}, j_! G) \cong \begin{cases} 0 & \text{for } n \neq 2 \\ G & \text{for } n = 2 \end{cases} \tag{0.2.5.1}$$

Proof. Put

$$V := \{x \in (\mathbb{P}^1_k)^{\text{rig}} \mid |x| \geq 1\}$$

and

$$S := V - (V \cap U) = (\mathbb{P}^1_k)^{\text{rig}} - U.$$

Then V is an affinoid admissible open subspace of $(\mathbb{P}^1_k)^{\text{rig}}$ and S is a closed constructible subset of V. $S^\sim_{\text{ét},V}$ is the closed subtopos of $V^\sim_{\text{ét}}$ complementary to the open subtopos $(U \cap V)^\sim_{\text{ét}}$ of $V^\sim_{\text{ét}}$. The composition of natural morphisms of sites

$$i : S_{\text{ét},V} \longrightarrow V_{\text{ét}} \longrightarrow (\mathbb{P}^1_k)^{\text{rig}}_{\text{ét}}$$

relations, $E = t(s^{-1}(W))$ is R-invariant. By quasicompactness, there is some quasicompact open subset $W' \subset X$ containing E. Now E is a pro-constructible and generalizing subset of the spectral space W'; this implies that E is an intersection of quasicompact open subsets. Let $Z = X \setminus W'$, which is a closed subset of X. Then $s^{-1}(Z) \subset R$ is closed, and in particular pro-constructible. As t is quasicompact, it follows that $T = t(s^{-1}(Z)) \subset X$ is pro-constructible. Thus, its closure $\bar{T} \subset X$ is the set of specializations of elements of T. We claim that $\bar{T}$ is R-invariant: Indeed, if $\xi \in \bar{T}$, there is some point $\xi' \in T$ specializing to ξ. If $r \in R$ is a point mapping under s to ξ, we can find $r' \in R$ mapping under s to ξ', as s is generalizing. Then $t(r)$ is a specialization of $t(r') \in T$, so $t(r) \in \bar{T}$. Finally, $U = X \setminus \bar{T}$ is an open R-invariant subset of X, such that $W \subset U \subset W'$. The inclusion $U \subset E$ does not hold with our definition of E, but we have $U \subset E'$ where $E' = t(s^{-1}(W'))$.

To see that X/R is T_0, start with two distinct points $\bar{x}, \bar{y} \in X/R$, and lift them to $x, y \in X$, so that their R-orbits are $R \cdot x = t(s^{-1}(x))$ and $R \cdot y = t(s^{-1}(y))$, respectively, and $R \cdot x \cap R \cdot y = \emptyset$. Note that $R \cdot x$ and $R \cdot y$ are both pro-constructible quasicompact subsets of X, as s is quasicompact and t is spectral; in particular, $R \cdot x$, $R \cdot y$ and $R \cdot x \cup R \cdot y$ are spectral spaces. We claim that it cannot happen that there are $x_1, x_2 \in R \cdot x$ and $y_1, y_2 \in R \cdot y$ such that x_1 generalizes to y_1 and x_2 specializes to y_2. Indeed, assume this was the case. As $R \cdot x \cup R \cdot y \subset X$ is a pro-constructible subset, it is a spectral space, and thus admits a maximal point; possibly changing the roles of x and y, we may assume that y is a maximal point of $R \cdot x \cup R \cdot y$. As $t : R \to X$ is generalizing, $(y, y_2) \in R$, and y_2 generalizes to x_2, we can find some point $(z, x_2) \in R$, where z is a generalization of y. Then $z \in R \cdot x \subset R \cdot x \cup R \cdot y$. As y is maximal in that space, it follows that $z = y$, but then $(y, x_2) \in R$, so that $R \cdot y = R \cdot x_2 = R \cdot x$, which is a contradiction.

Thus, we may assume that no point of $R \cdot x$ generalizes to a point of $R \cdot y$. In that case, we can find a quasicompact open subset $W \subset X$ containing x such that $W \cap R \cdot y = \emptyset$: Indeed, the intersection of $W \cap R \cdot y$ over all quasicompact open neighborhoods W of x is empty, as $R \cdot y$ contains no generalization of x. But then by quasicompactness $R \cdot y$ for the constructible topology, $W \cap R \cdot y = \emptyset$ for some such W. Now we run the above argument to find some open R-invariant subset $U \supset W$ of X. We claim that we can do this so that $U \cap R \cdot y = \emptyset$: This would finish the proof, as $R \cdot x \subset U$, while $U \cap R \cdot y = \emptyset$, so that the image of U in X/R separates x from y. To see that one can arrange $U \cap R \cdot y = \emptyset$, note that in the notation of the first paragraph, $E \cap R \cdot y = \emptyset$. As E is generalizing, one can for any point $e \in E$ find a quasicompact open neighborhood W'_e of e such that $W'_e \cap R \cdot y = \emptyset$; in other words, by quasicompactness of E we can choose W' so that $W' \cap R \cdot y = \emptyset$. In that case, also $E' \cap R \cdot y = \emptyset$, so as we have $U \subset E'$ for the R-invariant open subset $U \subset X$, we see that $U \cap R \cdot y = \emptyset$, as desired. □

Remark 2.8. Even if X is spectral, it can happen under the hypothesis of the lemma that the quotient space X/R is not spectral. In fact, using just profinite sets X and R, one can get any compact Hausdorff space as the quotient X/R: Namely, given a compact Hausdorff space Y, let X be the Stone-Cech compactification of Y considered as a discrete set. Then X is a profinite set which comes with a natural continuous surjective map to Y, and the induced equivalence relation $R \subset X \times X$ is closed, and thus profinite.

[Hue96]

Lemma 2.9. *Let X be a spectral space, and let $R \subset X \times X$ be a pro-constructible equivalence relation such that the maps $s, t : R \to X$ are generalizing. Assume that X/R has a basis for the topology given by open subsets whose preimages in X are quasicompact. Then X/R is a spectral space, and $X \to X/R$ is a spectral and generalizing map.*

(1.1.13) p. 44

Proof. Clearly, any specialization of a point in S lies in $\bar{S}$. Now let $x \in X$ be a point such that no generalization of x lies in S. Let $X_x \subset X$ be the set of generalizations of x, which is the intersection of all quasicompact open subsets U containing x; in particular, $X_x \subset X$ is a pro-constructible subset. Then

$$\emptyset = X_x \cap S = \bigcap_{U \ni x} (U \cap S) .$$

In other words, the $(X \setminus U) \cap S$ cover S. As S is quasicompact in the constructible topology, and $X \setminus U$ is constructible, it follows that there is some quasicompact open neighborhood U of x such that $(X \setminus U) \cap S = S$, i.e. $U \cap S = \emptyset$. Thus, x has an open neighborhood which does not meet S, so does not lie in the closure of S, as desired. □

[Hub 96] p. 2

All maps of analytic adic spaces are generalizing. This notably implies that surjective maps are quotient maps:

Lemma 2.5. *Let $f : Y \to X$ be a surjective and generalizing spectral map of spectral spaces. Then f is a quotient map.*

Proof. We have to show a subset $S \subset X$ is open if the preimage $T = f^{-1}(S) \subset Y$ is open, so assume that T is open. Note that in particular $Y \setminus T \subset Y$ is closed, and thus pro-constructible; therefore, its image $X \setminus S \subset X$ is pro-constructible. But as f is generalizing, this subset is also closed under specializations, and thus closed. Thus, S is open, as desired. □

We also recall the following classical lemma.

Lemma 2.6. *Let $f : S \to T$ be a continuous surjective map from a quasicompact space S to a compact Hausdorff space T. Then f is a quotient map.*

Proof. Note that for any $x \in T$, the open neighborhoods are cofinal with the closed neighborhoods. Assume that $V \subset T$ is a subset whose preimage $U = f^{-1}(V) \subset S$ is open with closed complement $Z = S \setminus U$, and let $x \in V$. Then

$$\emptyset = f^{-1}(x) \cap Z = \bigcap_{U_x \subset T} f^{-1}(\overline{U_x}) \cap Z .$$

Here, $U_x \subset T$ runs over open neighborhoods of x in T. The intersection is over closed subsets of S; by quasicompactness, there is some open neighborhood U_x of x such that $f^{-1}(\overline{U_x}) \cap Z = \emptyset$, i.e. $f^{-1}(\overline{U_x}) \subset U$. This implies that $f^{-1}(U_x) \subset U$, and thus $U_x \subset V$, as desired. □

[Hub 96] p. 2.

Def. 0.3.4 p. 9

Def. 0.4.2 p. 13

Here is a lemma about equivalence relations on spectral spaces. Recall that a topological space is quasiseparated if the intersection of any two quasicompact open subsets is quasicompact.

Lemma 2.7. *Let X be a quasiseparated locally spectral space, and let $R \subset X \times X$ be a pro-constructible equivalence relation such that the maps $s, t : R \to X$ are quasicompact and generalizing. Then the quotient space X/R is T_0. Moreover, for any quasicompact open $W \subset X$, there exists an open R-invariant subset $U \supset W$ such that $U \subset E$, where E is some R-invariant subset that is an intersection of a nonempty family of quasicompact open subsets.*

[Hub 96] p. 43, 44 non-pluricous

[Hub 96] (1.1.13) p. 44-45

Proof. We start with the second part, for which we follow the arguments from [Sta, Tag 0APA, 0APB]. Let $s, t : R \to X$ denote the two projections. Let $W \subset X$ be a quasicompact open subset. As s is quasicompact, $s^{-1}(W) \subset R$ is quasicompact open, and thus its image $E = t(s^{-1}(W)) \subset X$ is quasicompact, pro-constructible, and generalizing. Note that by general nonsense about equivalence

[Hub 96] Def. 1.1.4 p. 39, 38

[Hub 96] p. 20 here

(1.1.13) A topological space X is called *spectral* if X is quasi-compact and has a basis of quasi-compact open subsets, stable under finite intersections, and if every irreducible closed subset is the closure of a unique point ([Ho]). A topological space X is called *locally spectral* if every point of X has an open neighbourhood which is spectral. A locally spectral space X is spectral if and only if X is quasi-separated (i.e., the intersection of two quasi-compact open subsets of X is quasi-compact) and quasi-compact. A subset T of a locally spectral space X is called

(i) • *retro-compact* if, for every quasi-compact open subset U of X, the intersection $T \cap U$ is quasi-compact ([EGA*, 0.2.3.1])

(ii) • *globally constructible* if T is an element of the boolean algebra of subsets of X which is generated by the retro-compact open subsets of X ([EGA*, 0.2.3.2])

(iii) • *constructible* if every point of X has an open neighbourhood U such that $T \cap U$ is globally constructible in U ([EGA*, 0.2.3.10])

(iv) • *pro-constructible* (resp. *ind-constructible*) if every point of X has an open neighbourhood U such that $T \cap U$ is the intersection (resp. union) of constructible subsets of U ([EGA*, I.7.2.2])

(v) • *locally constructible* (resp. *locally pro-constructible*, resp. *locally ind-constructible*) if T is a constructible (resp. pro-constructible, resp. ind-constructible) subset of an open subset of X

(vi) • *convex* if, for every $x, y, z \in X$, the relations $x \succ y \succ z$ and $x, z \in T$ imply $y \in T$.

Let X be a quasi-compact locally spectral space, $(T_i)_{i \in I}$ a family of pro constructible subsets of X, and S an ind constructible subset of X such that $\bigcap_{i \in I} T_i \subseteq S$. Then there exists a finite subset J of I with $\bigcap_{i \in J} T_i \subseteq S$ ([Ho])

Let $f : X \to Y$ be a continuous mapping between locally spectral spaces. f is called

(i) • *quasi-separated* if, for every quasi-separated open subset V of Y, $f^{-1}(V)$ is quasi-separated

(ii) • *quasi-compact* if, for every quasi-compact open subset V of Y, $f^{-1}(V)$ is quasi-compact

(iii) • *spectral* if, for every quasi-compact quasi-separated open subsets $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, the restriction $f : U \to V$ is quasi-compact.

If f is quasi-compact and spectral then, for every pro-constructible subset T of X, $f(T)$ is pro-constructible in Y.

(1.1.14) The following simple properties of valuations are used freely.

a) Let $v: K \longrightarrow \Gamma \cup \{0\}$ be a valuation of a field K and let $A \subseteq K$ be the valuation ring of v, i.e. $A = \{k \in K \mid v(k) \leq 1\}$. Then A is integrally closed in K. For every convex subgroup H of Γ, the set $\mathfrak{p}_H := \{k \in K \mid v(k) < H\}$ is a prime ideal of A. By $H \mapsto \mathfrak{p}_H$ we get a bijection from the set of all convex subgroups of Γ onto the set of all prime ideals of A. The set of prime ideals of A is totally ordered.

b) Let A be a valuation ring of a field K. For every prime ideal $\mathfrak{p}$ of A, the localization $A_\mathfrak{p}$ is a valuation ring of K and $\mathfrak{p}$ is the maximal ideal of $A_\mathfrak{p}$. By $\mathfrak{p} \mapsto A_\mathfrak{p}$ we get a bijection from the set of all prime ideals of A onto the set of all valuation rings of K which contain A. If $v: K \longrightarrow \Gamma \cup \{0\}$ is a valuation which corresponds to A and if $H \subseteq \Gamma$ is the convex subgroup which corresponds to $\mathfrak{p}$ then the mapping $K \to \Gamma/H \cup \{0\}$, $k \mapsto v(k) \mathrm{mod} H$ is a valuation of K which corresponds to the valuation ring $A_\mathfrak{p}$.

c) Let A be a valuation ring of a field K, and let $\lambda: A \to k$ be the natural mapping from A onto its residue class field. For every valuation ring B of k, $\lambda^{-1}(B)$ is a valuation ring of K. By $B \mapsto \lambda^{-1}(B)$ we get a bijection form the set of all valuation rings of k onto the set of all valuation rings of K which are contained in A.

d) Let K be a field and $A \subseteq K$ a local ring. Then there exists a valuation ring B of K which dominates A.

e) Let $K_1 \hookrightarrow K_2$ be an extension of fields. Let A_2 be a valuation ring of K_2 and put $A_1 := A_2 \cap K_1$. For $i = 1, 2$, let G_i be the set of all valuation rings of K_i which contain A_i and let S_i be the set of all valuation rings of K_i which are contained in A_i. Then the mappings $\varphi: G_2 \to G_1$, $B \mapsto B \cap K_1$ and $\psi: S_2 \to S_1$, $B \mapsto B \cap K_1$ are surjective (obvious by (a) – (d)). If K_2 is algebraic over K_1 then φ is bijective, the quotient Γ_2/Γ_1 of the value groups of A_2 and A_1 is a torsion group and the extension $k_1 \hookrightarrow k_2$ of the residue class fields of A_1 and A_2 is algebraic.

f) Let $K \hookrightarrow K_1$ and $K \hookrightarrow K_2$ be extensions of fields, and let A_1, A_2 be valuation rings of K_1, K_2 with $A_1 \cap K = A_2 \cap K$. Then there exist a composite K_3 of K_1 and K_2 over K and a valuation ring A_3 of K_3 with $A_1 = A_3 \cap K_1$ and $A_2 = A_3 \cap K_2$.

g) Let A be a valuation ring of a field K with corresponding valuation $v: K \to \Gamma \cup \{0\}$. The set of all sets of the form $\{x \in K \mid v(x) < \gamma\}$ $(\gamma \in \Gamma)$ is a fundamental system of neighbourhoods of 0 of a field topology of K (the valuation topology of A). The completion $\hat{K}$ of K is a field, the completion $\hat{A}$ of A is a valuation ring of $\hat{K}$, and the topology of $\hat{K}$ is the valuation topology of $\hat{A}$. The value groups and the residue class fields of A and $\hat{A}$ are equal. For every valuation ring B of K with $A \subseteq B \subsetneq K$, the valuation topologies of A and B on K coincide.

1.2 Morphisms of finite type

A ring homomorphism $f: A \to B$ between f.adic rings is called *adic* if there are rings of definition A_0, B_0 of A, B and an ideal of definition I of A_0 such that $f(A_0) \subseteq B_0$ and $f(I) \cdot B_0$ is an ideal of definition of B_0. (Then, for all rings of definition A_0, B_0 of A, B with $f(A_0) \subseteq B_0$ and every ideal of definition I of A_0, $f(I) \cdot B_0$ is an ideal

Proof. The set of open subsets of X/R whose preimages in X are quasicompact is stable under finite intersections. To see that X/R is spectral, it remains to see that every irreducible closed subset $Z \subset X/R$ has a unique generic point. Uniqueness holds because X/R is T_0 by Lemma 2.7. For existence, look at the cofiltered intersection of all open subsets $U \subset X/R$ with $U \cap Z \neq \emptyset$ whose preimage in X is quasicompact. We need to see that this intersection is nonempty. But the preimages of such U are quasicompact open subsets, thus compact Hausdorff for the constructible topology. As they are all nonempty, their intersection is nonempty. Thus, the intersection of the U's is nonempty as well, as desired.

As X/R has a basis of opens for which the preimage in X is quasicompact, it follows that $X \to X/R$ is spectral. It remains to see that $X \to X/R$ is generalizing. Assume that $\bar{x}$ generalizes to $\bar{x}' \in X/R$. We claim first that there is some way to lift this to a generalization x to x' in X. If not, then the final paragraph of the proof of Lemma 2.7 would produce an open R-invariant neighborhood of x in X which does not contain x', contradicting that $\bar{x}$ generalizes to $\bar{x}'$. Now if $x_0 \in X$ is any lift of $\bar{x}$, then $(x_0, x) \in R$. But $t : R \to X$ is generalizing, and x' is a generalization of x in X, so we can find a generalization $(x_0', x') \in R$ of (x_0, x). But then $x_0' \in X$ is a generalization of x_0 mapping to $\bar{x}' \in X/R$, as desired. □

Lemma 2.10. *In the situation of Lemma 2.7, assume that $R \to X$ is open. Then the quotient space X/R is locally spectral and quasiseparated, and $X \to X/R$ is an open spectral qcqs map.*

Proof. For any quasicompact open subset $U \subset X$, the set $t(s^{-1}(U)) \subset X$ is a quasicompact R-invariant open subset under the assumptions. Note that $t(s^{-1}(U)) \subset X$ is also the preimage of the image of U in X/R, so that $X \to X/R$ is open. For the other assertions, replacing X by U, we can assume that X is quasicompact, thus spectral. We claim that the quasicompact open subsets $V_0 \subset X/R$ whose preimages in X are quasicompact, form a basis for the topology. Thus, let $W_0 \subset X/R$ be open, with preimage $W \subset X$ an R-invariant open subset. Pick any $x_0 \in W_0$, and lift it to $x \in W$. Then x has a quasicompact open neighborhood $x \in V' \subset W$, and $V = t(s^{-1}(V')) \subset W$ is a quasicompact R-invariant open subset. Its image $V_0 \subset W_0$ thus contains x_0, is quasicompact, with inverse image $V \subset X$ still quasicompact.

It remains to see that every irreducible closed subset $\bar{Z} \subset X/R$ has a generic point. Equivalently, the intersection of all quasicompact open subsets $V_0 \subset X/R$ with $V_0 \cap \bar{Z} \neq \emptyset$ is nonempty. But the corresponding intersection of the preimages in X is nonempty by compactness of the constructible topology, so the result follows. □

We will need the following result about inverse limits.

Lemma 2.11. *Let X_i, $i \in I$, be a cofiltered inverse system of spectral spaces along spectral maps, with inverse limit $X = \varprojlim_i X_i$. Then X is a spectral space, the maps $X \to X_i$ are spectral, and a map $Y \to X$ from a spectral space is spectral if and only if all composites $Y \to X \to X_i$ are spectral. Moreover, if Y is a spectral space with a spectral map $Y \to X$ such that the composite maps $Y \to X_i$ are generalizing, then $Y \to X$ is generalizing. In particular, if $Y \to X$ is in addition surjective, then by Lemma 2.5 it is a quotient map.*

Proof. The first part follows for example from the interpretation of the category of spectral spaces with spectral maps as the pro-category of finite T_0-spaces. Now assume $Y \to X$ is a spectral map of spectral spaces such that $Y \to X_i$ is generalizing for all $i \in I$. Let $y \in Y$ map to $x \in X$, and let $\tilde{x} \in X$ be a generalization of $x \in X$. We look for a generalization $\tilde{y}$ of Y mapping to $\tilde{x}$. For this, we may replace Y by its localization Y_y at y (i.e. the set of generalizations of y). Let $x_i \in X_i$

be the image of x, and $\widetilde{x}_i \in X_i$ be the image of $\widetilde{x}$. Then, for all $i \in I$, $\widetilde{x}_i$ is a generalization of x_i. As $Y \to X_i$ is generalizing, the preimage of $\widetilde{x}_i$ in Y is a non-empty pro-constructible subset of Y. For varying i, these preimages form a cofiltered inverse system of nonempty pro-constructible subsets of Y; their intersection is thus nonempty (by Tychonoff). Any point in the intersection is a generalization of y mapping to $\widetilde{x}$, as desired. □

3. Perfectoid Spaces

Let p be a fixed prime throughout. Recall that a topological ring R is Tate if it contains and open and bounded subring $R_0 \subset R$ and a topologically nilpotent unit $\varpi \in R$; such elements are called pseudo-uniformizers. Let $R^\circ \subset R$ denote the set of powerbounded elements. If $\varpi \in R$ is a pseudo-uniformizer, then necessarily $\varpi \in R^\circ$. Furthermore, if $R^+ \subset R$ is a ring of integral elements, i.e. an open and integrally closed subring of R°, then $\varpi \in R^+$. Indeed, since $\varpi^n \to 0$ as $n \to \infty$ and since R^+ is open, we have $\varpi^n \in R^+$ for some $n \geq 1$. Since R^+ is integrally closed, $\varpi \in R^+$.

The following definition is due to Fontaine, [Fon13].

Definition 3.1. *A Tate ring R is* perfectoid *if R is complete, uniform, i.e. $R^\circ \subset R$ is bounded, and there exists a pseudo-uniformizer $\varpi \in R$ such that $\varpi^p | p$ in R° and the Frobenius map*

$$\Phi \colon R^\circ/\varpi \to R^\circ/\varpi^p : x \mapsto x^p$$

is an isomorphism.

Hereafter we use the following notational convention. If R is a ring, and $I, J \subset R$ are ideals containing p such that $I^p \subset J$, then $\Phi \colon R/I \to R/J$ will refer to the ring homomorphism $x \mapsto x^p$.

Remark 3.2. The condition that $\Phi : R^\circ/\varpi \to R^\circ/\varpi^p$ is an isomorphism is independent of the choice of a pseudo-uniformizer $\varpi \in R$ such that $\varpi^p | p$. In fact, we claim that it is equivalent to the condition that $\Phi : R^\circ/p \to R^\circ/p$ is surjective.

Indeed, for any complete Tate ring R and pseudo-uniformizer ϖ satisfying $\varpi^p | p$ in R°, the Frobenius map $\Phi \colon R^\circ/\varpi \to R^\circ/\varpi^p$ is necessarily injective. Indeed, if $x \in R^\circ$ satisfies $x^p = \varpi^p y$ for some $y \in R^\circ$ then the element $x/\varpi \in R$ lies in R° since its p-th power does. Thus, the isomorphism condition on Φ in Definition 3.1 is equivalent to surjectivity of $\Phi : R^\circ/\varpi \to R^\circ/\varpi^p$. Thus, if $\Phi : R^\circ/p \to R^\circ/p$ is surjective, then R is perfectoid. Conversely, assume that R is perfectoid. Let $x \in R^\circ$ be any element. By using that $\Phi : R^\circ/\varpi \to R^\circ/\varpi^p$ is surjective and successive ϖ^p-adic approximation, we can write

$$x = x_0^p + \varpi^p x_1^p + \varpi^{2p} x_2^p + \dots ,$$

where $x_i \in R^\circ$. But then

$$x \equiv (x_0 + \varpi x_1 + \varpi^2 x_2 + \dots)^p \mod p ,$$

showing that $\Phi : R^\circ/p \to R^\circ/p$ is surjective, as desired.

Remark 3.3. The field $\mathbf{Q}_p$ is not perfectoid, even though the Frobenius map on $\mathbf{F}_p$ is an isomorphism. The issue is that there is no element $\varpi \in \mathbf{Z}_p$ whose p-th power divides p. More generally, a discretely valued non-archimedean field K cannot be perfectoid. Indeed, if ϖ is a pseudo-uniformizer as in Definition 3.1, then ϖ is a non-zero element of the maximal ideal, so the quotients K°/ϖ and K°/ϖ^p are Artin local rings of different lengths and hence cannot be isomorphic.

Example 3.4. (i) The cyclotomic field $\mathbf{Q}_p^{\mathrm{cycl}}$, the completion of the cyclotomic extension $\mathbf{Q}_p(\mu_{p^\infty})$.

(ii) The t-adic completion of $\mathbf{F}_p((t))(t^{1/p^\infty})$, which we will write as $\mathbf{F}_p((t^{1/p^\infty}))$.

(iii) The algebra $\mathbf{Q}_p^{\mathrm{cycl}}\langle T^{1/p^\infty}\rangle$. This is defined as $A[1/p]$, where A is the p-adic completion of $\mathbf{Z}_p^{\mathrm{cycl}}[T^{1/p^\infty}]$.

(iv) As an example of a perfectoid Tate ring which does not live over a field, one may take

$$R = \mathbf{Z}_p^{\mathrm{cycl}}[[T^{1/p^\infty}]]\langle (p/T)^{1/p^\infty}\rangle[1/T] .$$

Here we can take $\varpi = T^{1/p}$, because $\varpi^p = T$ divides p in R°.

Proposition 3.5. *Let R be a topological ring with $pR = 0$. The following are equivalent:*

(1) The topological ring R is a perfectoid Tate ring.

(2) The topological ring R is a perfect complete Tate ring.

Of course, perfect means that $\Phi\colon R \to R$ is an isomorphism.

Proof. Let R be a complete Tate ring. If R is perfect, then take ϖ any pseudo-uniformizer. The condition $\varpi^p | p = 0$ is vacuous. If $x \in R$ is powerbounded, then so is x^p, and vice versa, which means that $\Phi\colon R^\circ \to R^\circ$ is an isomorphism. This shows that $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$ is surjective, and we had seen in Remark 3.2 that injectivity is automatic. It remains to see that R is automatically uniform. For this, take any open and bounded subring $R_0 \subset R$. As $\Phi : R \to R$ is an isomorphism, which is automatically open by Banach's open mapping theorem, there is some n such that $\varpi^n R_0 \subset \Phi(R_0)$. Equivalently, $\Phi^{-1}(R_0) \subset \varpi^{-n/p} R_0$. This implies that

$$\Phi^{-2}(R_0) \subset \Phi^{-1}(\varpi^{-n/p} R_0) \subset \varpi^{-n/p-n/p^2} R_0$$

and inductively

$$\Phi^{-k}(R_0) \subset \varpi^{-n/p-n/p^2-\ldots-n/p^k} R_0 .$$

In particular, for all $k \geq 0$,

$$\Phi^{-k}(R_0) \subset \varpi^{-n} R_0 .$$

On the other hand, $R^\circ \subset \bigcup_{k\geq 0} \Phi^{-k}(R_0)$, and so $R^\circ \subset \varpi^{-n} R_0$ is bounded, as desired.

Conversely, if R is perfectoid, then $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$ is an isomorphism, and therefore so is $R^\circ/\varpi^n \to R^\circ/\varpi^{np}$ by induction. Taking inverse limits and using completeness, we find that $\Phi\colon R^\circ \to R^\circ$ is an isomorphism. Inverting ϖ shows that R is perfect. □

Definition 3.6. *A perfectoid field is a perfectoid Tate ring R which is a nonarchimedean field.*

Remark 3.7. It is not clear a priori that a perfectoid ring which is a field is a perfectoid field. However, this has recently been answered affirmatively by Kedlaya, [Ked16].

One easily checks the following equivalent characterization, which is the original definition, [Sch12], [KL15].

Proposition 3.8. *Let K be a nonarchimedean field. Then K is a perfectoid field if and only if the following conditions hold:*

(i) *The nonarchimedean field K is not discretely valued,*

(ii) *the absolute value $|p| < 1$, and*

(iii) *the Frobenius $\Phi\colon \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective.*

□

Next, we recall the process of tilting.

Definition 3.9. *Let R be a perfectoid Tate ring. The* tilt *of R is the topological ring*

$$R^\flat = \varprojlim_{x \mapsto x^p} R\,,$$

with the inverse limit topology, the pointwise multiplication, and the addition given by

$$(x^{(0)}, x^{(1)}, \dots) + (y^{(0)}, y^{(1)}, \dots) = (z^{(0)}, z^{(1)}, \dots)$$

where

$$z^{(i)} = \lim_{n \to \infty} (x^{(i+n)} + y^{(i+n)})^{p^n} \in R\,.$$

Lemma 3.10. *The limit $z^{(i)}$ above exists and defines a ring structure making $R^\flat$ a perfectoid $\mathbf{F}_p$-algebra. The subset $R^{\flat\circ}$ of power-bounded elements is given by the topological ring isomorphism*

$$R^{\flat\circ} = \varprojlim_{x \mapsto x^p} R^\circ \cong \varprojlim_{\Phi} R^\circ/\varpi\,,$$

where $\varpi \in R$ is a pseudo-uniformizer which divides p in R°.

Furthermore, there exists a pseudo-uniformizer $\varpi \in R$ with $\varpi^p | p$ in R° which admits a sequence of p-power roots ϖ^{1/p^n}, giving rise to an element $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots) \in R^{\flat\circ}$, which is a pseudo-uniformizer of $R^\flat$. Then $R^\flat = R^{\flat\circ}[1/\varpi^\flat]$.

Proof. Let ϖ_0 be a pseudo-uniformizer of R such that $\varpi_0 | p$ in R°. Let us check that the map

$$\varprojlim_{x \mapsto x^p} R^\circ \to \varprojlim_{\Phi} R^\circ/\varpi_0$$

is an isomorphism. We have to see that any sequence $(\overline{x}_0, \overline{x}_1, \dots) \in \varprojlim_\Phi R^\circ/\varpi_0$ lifts uniquely to a sequence $(x_0, x_1, \dots) \in \varprojlim_\Phi R^\circ$. This lift is given by $x^{(i)} = \lim_{n\to\infty} x_{n+i}^{p^n}$, where $x_j \in R^\circ$ is any lift of $\overline{x}_j$. (For the convergence of that limit, note that if $x \equiv y \pmod{\varpi_0^n}$, then $x^p \equiv y^p \pmod{\varpi_0^{n+1}}$.) This shows that we get a well-defined ring

$$\varprojlim_{x \mapsto x^p} R^\circ \subset \varprojlim_{x \mapsto x^p} R = R^\flat\,,$$

which agrees with the powerbounded elements $R^{\flat\circ} \subset R^\flat$.

Next, we construct the element $\varpi^\flat$. For this, we assume that $\varpi_0 = \varpi_1^p$ for some pseudouniformizer $\varpi_1 \in R$ such that $\varpi_0 = \varpi_1^p | p$. Any preimage of ϖ_1 under $R^{\flat\circ} = \varprojlim_\Phi R^\circ/\varpi_0 \to R^\circ/\varpi_0$ is an element $\varpi^\flat$ with the right properties. It is congruent to ϖ_1 modulo ϖ_0, and therefore it is also a topologically nilpotent, and invertible in $R^\flat$. Then $\varpi = \varpi^{\flat\sharp}$ is the desired pseudo-uniformizer of R°.

Now one sees that $R^\flat = R^{\flat\circ}[1/\varpi^\flat]$ as multiplicative monoids, and in fact as rings. This implies that the addition in $R^\flat$ is always well-defined, and defines a ring of characteristic p, which is perfect by design. The rest of the lemma follows easily. □

We have a continuous, multiplicative (but not additive) map $R^\flat = \varprojlim_{x \mapsto x^p} R \to R$ by projecting onto the zeroth coordinate; call this $f \mapsto f^\sharp$. This projection defines a ring isomorphism $R^{\flat\circ}/\varpi^\flat \cong R^\circ/\varpi$. The open and integrally closed subrings of $R^{\flat\circ}$ and R° correspond exactly to the integrally closed subrings of their common quotients modulo $\varpi^\flat$ and ϖ. This defines an inclusion-preserving

$$\mathcal{O}_{A,x} = \varinjlim_{\substack{x \in U \\ U \text{ rational}}} \mathcal{O}_A(U) . \tag{1}$$

For every rational subset U of $\mathrm{Spa}A$ with $x \in U$ the valuation $x: A^\flat \to \Gamma_x \cup \{0\}$ extends uniquely to a continuous valuation $v_U: \mathcal{O}_A(U) = F_A(U)^\flat \to \Gamma_x \cup \{0\}$ (cf. (1.5.ii)). Then by (1), the valuations v_U define a valuation

$$v_x: \mathcal{O}_{A,x} \to \Gamma_x \cup \{0\} .$$

Thus we have a triple $(\mathrm{Spa}A, \mathcal{O}_A, (v_x | x \in \mathrm{Spa}A))$ consisting of a topological space $\mathrm{Spa}A$, a presheaf $\mathcal{O}_A$ of complete topological rings on $\mathrm{Spa}A$ and a family of valuations v_x on the stalks $\mathcal{O}_{A,x}$. For every open subset U of $\mathrm{Spa}A$ we put

$$\mathcal{O}_A^+(U) = \{ f \in \mathcal{O}_A(U) \mid v_x(f) \leqq 1 \text{ for all } x \in U \} .$$

Then $\mathcal{O}_A^+$ is a presheaf of rings on $\mathrm{Spa}A$. For every $x \in \mathrm{Spa}A$ let $\mathcal{O}_{A,x}^+$ denote the stalk of $\mathcal{O}_A^+$ at x.

Proposition 1.6 (i) *For every $x \in \mathrm{Spa}A$, the stalk $\mathcal{O}_{A,x}$ is a local ring and the maximal ideal of $\mathcal{O}_{A,x}$ is the support of v_x.*
(ii) *For every $x \in \mathrm{Spa}A$ the stalk $\mathcal{O}_{A,x}^+$ is a local ring. We have $\mathcal{O}_{A,x}^+ = \{ f \in \mathcal{O}_{A,x} \mid v_x(f) \leqq 1 \}$ with maximal ideal $\{ f \in \mathcal{O}_{A,x} \mid v_x(f) < 1 \}$.*
(iii) *For every open subset U of $\mathrm{Spa}A$ and every $f, g \in \mathcal{O}_A(U)$ the set $\{x \in U | v_x(f) \leqq v_x(g) \neq 0\}$ is open in $\mathrm{Spa}A$.*
(iv) *For every rational subset U of $\mathrm{Spa}A$ we have $\mathcal{O}_A(U) = F_A(U)^\flat$ and $\mathcal{O}_A^+(U) = F_A(U)^+$.*

Proof. (i) Let x be a point of $\mathrm{Spa}A$, U an open subset of $\mathrm{Spa}A$ with $x \in U$ and f an element of $\mathcal{O}_A(U)$ with $v_x(f) \neq 0$. We have to show that f is a unit in $\mathcal{O}_{A,x}$. Let W be a rational subset of $\mathrm{Spa}A$ with $x \in W \subseteq U$. The valuation v_x of $\mathcal{O}_{A,x}$ gives by restriction to $\mathcal{O}_A(W)$ a continuous valuation v_W of $\mathcal{O}_A(W)$. Since $v_W(f) \neq 0$, there exists a finite subset T of the f-adic ring $F_A(W)^\flat = \mathcal{O}_A(W)$ such that $T \cdot F_A(W)^\flat$ is open in $F_A(W)^\flat$ and $v_W(t) \leqq v_W(f)$ for every $t \in T$. Then we have in $\mathrm{Spa}F_A(W)$ the rational subset $V = R\left(\frac{T}{f}\right)$ with $v_W \in V$, and f is a unit in $F_B(V)^\flat$ where $B := F_A(W)$. Now (1.5.ii) and (1.5.iii) imply that there exists a rational subset S of $\mathrm{Spa}A$ such that $x \in S \subseteq W$ and f is a unit in $\mathcal{O}_A(S)$.
(ii) With (iii) we obtain $\mathcal{O}_{A,x}^+ = \{ f \in \mathcal{O}_{A,x} \mid v_x(f) \leqq 1 \}$. Then (i) implies that $\mathcal{O}_{A,x}^+$ is a local ring with maximal ideal $\{ f \in \mathcal{O}_{A,x} \mid v_x(f) < 1 \}$.
(iii) We can assume that U is rational. The set $\{v \in \mathrm{Spa}F_A(U) \mid v(f) \leqq v(g) \neq 0\}$ is open in $\mathrm{Spa}F_A(U)$ by the definition of the topology of $\mathrm{Spa}F_A(U)$. Now the assertion follows from (1.5.ii).
(iv) By definition we have $\mathcal{O}_A(U) = F_A(U)^\flat$. By (1.5.ii) the mapping $\mathrm{Spa}F_A(U) \to \mathrm{Spa}A$ is injective with image U. Hence $\mathcal{O}_A^+(U) = \{ f \in F_A(U)^\flat \mid v(f) \leqq 1 \text{ for all } v \in \mathrm{Spa}F_A(U) \}$. Now [H1, 3.3.i] implies $\mathcal{O}_A^+(U) = F_A(U)^+$. □

In general, the presheaf $\mathcal{O}_A$ is not a sheaf of rings as the following example of M. Rost shows. Let $A^\flat = \mathbb{Z}[X_1, X_2, X_3]_{X_1 X_2}$ be the localization of the

cal ring homomorphism $h: A \to A\left\langle \frac{T}{s} \right\rangle$. We factorize h into $A \xrightarrow{f} A\left(\frac{T}{s}\right) \xrightarrow{i} A\left\langle \frac{T}{s} \right\rangle$. Since f and i are adic, h is adic, and hence $g^{-1}(V)$ is a rational subset of $\mathrm{Spa}A\left\langle \frac{T}{s} \right\rangle$ for every rational subset V of $\mathrm{Spa}A$ (by [H1, 3.8.iv]). By [H1, 3.9] $\mathrm{Spa}(i) : \mathrm{Spa}A\left\langle \frac{T}{s} \right\rangle \to \mathrm{Spa}A\left(\frac{T}{s}\right)$ is a homeomorphism and maps rational subsets to rational subsets. So it remains to show that $\mathrm{Spa}(f): \mathrm{Spa}A\left(\frac{T}{s}\right) \to \mathrm{Spa}A$ is a homeomorphism from $\mathrm{Spa}A\left(\frac{T}{s}\right)$ onto U and maps rational subsets to rational subsets. It is evident from our definition of $A\left(\frac{T}{s}\right)$ that $\mathrm{Spa}(f)$ is a homeomorphism from $\mathrm{Spa}A\left(\frac{T}{s}\right)$ onto U. Let V be a rational subset of $\mathrm{Spa}A\left(\frac{T}{s}\right)$. We choose $\ell, g_1, \ldots, g_n \in A\left(\frac{T}{s}\right)^{\triangleright} = A^{\triangleright}\left[\frac{1}{s}\right]$ such that $V = \left\{ v \in \mathrm{Spa}A\left(\frac{T}{s}\right) \mid v(g_i) \leqq v(\ell) \neq 0 \text{ for } i = 1, \ldots, n \right\}$. Multiplying $\ell, g_1, \ldots, g_n$ with a suitable power of s, we may assume that there exists an element d of $A^{\triangleright}$ and a finite subset C of $A^{\triangleright}$ with $\ell = f(d)$ and $\{g_1, \ldots, g_n\} = f(C)$. Since V is quasi-compact, there exists by [H1, 3.11] a neighbourhood E of 0 in $A^{\triangleright}$ with $v(f(e)) \leqq v(\ell)$ for every $v \in V$, $e \in E$. Let D be a finite subset of E such that $D \cdot A^{\triangleright}$ is open in $A^{\triangleright}$. Then we have the rational subset $W = R\left(\frac{C \cup D}{d}\right)$ of $\mathrm{Spa}A$, and $\mathrm{Spa}(f)(V) = U \cap W$. Hence $\mathrm{Spa}(f)(V)$ is rational in $\mathrm{Spa}A$.

(iii) The existence and the uniqueness of r follow from (1.3.i). By (i) there exists a unique continuous ring homomorphism $h: B \to F_A(V)$ with $h_{A,V} = h \circ h_{A,U}$. We have $\mathrm{im}(\mathrm{Spa}(h)) \subseteq g^{-1}(V)$ (since $\mathrm{im}(\mathrm{Spa}(h_{A,V})) \subseteq V$). Then by (1.3.i) there exists a unique continuous ring homomorphism $f: F_B(g^{-1}(V)) \to F_A(V)$ with $h = f \circ h_{B,\, g^{-1}(V)}$. The ring homomorphism f is the inverse of r. □

Let A be an affinoid ring. For every rational subset W of $\mathrm{Spa}\,A$ we have the affinoid ring $F_A(W) = (F_A(W)^{\triangleright}, F_A(W)^{+})$. If $V \subseteq W$ then we have by (1.5.i) a canonical continuous ring homomorphism $\rho_{V,W}: F_A(W)^{\triangleright} \to F_A(V)^{\triangleright}$. For every open subset U of $\mathrm{Spa}A$ we put

$$\mathcal{O}_A(U) = \varprojlim_{V} F_A(V)^{\triangleright},$$

where the projective limit is taken over all rational subsets V of $\mathrm{Spa}A$ contained in U and with respect to the ring homomorphisms $\rho_{V,W}$. We equip $\mathcal{O}_A(U)$ with the projective limit topology. Then $\mathcal{O}_A(U)$ is a complete topological ring. Furthermore, if V, W are open subsets of $\mathrm{Spa}A$ with $V \subseteq W$ then we have a canonical continuous ring homomorphism $\mathcal{O}_A(W) \to \mathcal{O}_A(V)$. Thus we have a presheaf $\mathcal{O}_A$ on $\mathrm{Spa}A$ with values in the category of complete topological rings.

For every $x \in \mathrm{Spa}A$ let $\mathcal{O}_{A,\,x} = \varinjlim_{x \in U} \mathcal{O}_A(U)$ be the stalk of $\mathcal{O}_A$ at x. (The inductive limit is taken in the category of rings.) Since the rational subsets of $\mathrm{Spa}A$ form a basis of the topology of $\mathrm{Spa}A$, we have

$A^\triangleright\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$ should be an element of $\mathcal{O}_A(U)$. So we will define a ring topology on $A^\triangleright\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$, and then define $\mathcal{O}_A(U)$ to be the completion of $A^\triangleright\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$.

Lemma and definition. (i) *Let A be a f-adic ring, $s_1,\ldots,s_n$ elements of A and $T_1,\ldots,T_n$ subsets of A such that $T_i \cdot A$ is open in A for $i = 1,\ldots,n$. Let A_0 be a ring of definition of A and let I be an ideal of definition of A_0. Let T be the group topology of the localization $A_{s_1\ldots s_n} = A\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$ such that $\{I^n \cdot B \mid n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0, where B is the subring $A_0\left[\frac{t}{s_i} | i = 1,\ldots,n, t \in T_i\right]$ of $A_{s_1\ldots s_n}$. Then T is even a ring topology and independent of the choice of A_0 and I. The topological ring $(A_{s_1\ldots s_n}, T)$ is denoted by $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$. The completion of $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ is denoted by $A\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$.*

(ii) *Let $A = (A^\triangleright, A^+)$ be an affinoid ring, $s_1,\ldots,s_n$ elements of $A^\triangleright$ and $T_1,\ldots,T_n$ subsets of $A^\triangleright$ such that $T_i \cdot A^\triangleright$ is open in $A^\triangleright$ for $i = 1,\ldots,n$. Let C be the integral closure of $A^+\left[\frac{t}{s_i} \mid i = 1,\ldots,n, t \in T_i\right]$ in $A^\triangleright\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$. Then C is a ring of integral elements of $B := A^\triangleright\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$. The affinoid ring (B,C) is denoted by $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$. The completion of $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ is denoted by $A\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$.*

Proof. (i) By the subsequent Lemma 1.1 the set $T_i \cdot A_0$ is open in A, i.e., there exists a $n \in \mathbb{N}$ with $I^n \subseteq T_i \cdot A_0$. Then $\frac{t}{s_i} \cdot I^n \subseteq A_0\left[\frac{t}{s_i} \mid t \in T_i\right]$. Hence, for every $x \in A\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$ and every neighbourhood U of 0 in the topology T, there exists a neighbourhood V of 0 in T with $x \cdot V \subseteq U$. Consequently T is a ring topology. Let A_1, A_2 be two rings of definition of A and I_1, I_2 ideals of definition of A_1, A_2. For every $m \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ with $I_1^k \subseteq I_2^m$, and then $I_1^k \cdot A_1\left[\frac{t}{s_i} \mid i = 1,\ldots,n, t \in T_i\right] \subseteq I_2^m \cdot A_2\left[\frac{t}{s_i} \mid i = 1,\ldots,n, t \in T_i\right]$. Hence T is independent of the choice of A_0 and I. □

Lemma 1.1 *Let A be a f-adic ring and T a subset of A such that $T \cdot A$ is open in A. Then, for every $n \in \mathbb{N}$ and every neighbourhood G of 0 in A, the set $T^n \cdot G$ is open in A.*

Proof. With $T \cdot A$ also $T^n \cdot A = (T \cdot A)^n$ is open. Let U be a subset of A and S a finite subset of U such that $\{U^m | m \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 in A, $S \cdot U = U^2 \subseteq U$ and $U \subseteq T^n \cdot A$. We choose a finite subset R of A with $S \subseteq T^n \cdot R$, and a $k \in \mathbb{N}$ with $R \cdot U^k \subseteq G$. Then $U^{k+1} = S \cdot U^k \subseteq (T^n \cdot R) \cdot U^k = T^n \cdot (R \cdot U^k) \subseteq T^n \cdot G$. □

Our definition of $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ immediately implies

Notations

This paper is based on [H1]. We recall some notations from [H1]. For more details and all unexplained notations we refer to [H1].

All our valuations are multiplicative. For a valuation $v: A \to \Gamma \cup \{0\}$ of a ring A the prime ideal $\operatorname{supp}(v) := \{a \in A \mid v(a) = 0\}$ of A is called the support of A.

The group of units of a ring A is denoted by A^*, and if R, S are subsets of A then $R \cdot S$ denotes the additive subgroup of A generated by $\{r \cdot s \mid r \in R, s \in S\}$.

For a topological ring A, the set of power-bounded elements of A is denoted by A° and the set of topologically nilpotent elements of A is denoted by $A^{\circ\circ}$. A ring of definition of a f-adic ring A is an open adic subring of A. A Tate ring is a f-adic ring which has a topologically nilpotent unit. An affinoid ring is a pair $A = (A^\triangleright, A^+)$ where $A^\triangleright$ is a f-adic ring and A^+ is a ring of integral elements of $A^\triangleright$, i.e. A^+ is a subring of $A^\triangleright$ which is open and integrally closed in $A^\triangleright$ and is contained in $(A^\triangleright)^\circ$. For an affinoid ring $A = (B, C)$, the pair of the completions $\hat{A} := (\hat{B}, \hat{C})$ is an affinoid ring which is called the completion of A. (In this paper complete always means complete and hausdorff). For an affinoid ring A, we put $\operatorname{Spa}A = \{v \mid v$ is a continuous valuation of $A^\triangleright$ with $v(a) \leqq 1$ for every $a \in A^+\}$ and equip $\operatorname{Spa}A$ with the topology generated by the sets $\{v \in \operatorname{Spa}A \mid v(a) \leqq v(b) \neq 0\}$ $(a, b \in A^\triangleright)$. A subset U of $\operatorname{Spa}A$ is called rational if there are elements $s_1, \ldots, s_n$ of $A^\triangleright$ and finite subsets $T_1, \ldots, T_n$ of $A^\triangleright$ such that $T_i \cdot A^\triangleright$ is open in $A^\triangleright$ for $i = 1, \ldots, n$ and $U = \bigcap_{i=1}^n \{v \in \operatorname{Spa}A \mid v(t) \leqq v(s_i) \neq 0$ for all $t \in T_i\}$. The rational subsets of $\operatorname{Spa}A$ form a basis of the topology of $\operatorname{Spa}A$ and are quasi-compact.

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1 A presheaf on $\operatorname{Spa}A$

In this section we define, for every affinoid ring $A = (A^\triangleright, A^+)$, a presheaf $\mathcal{O}_A$ of "analytic functions" on the topological space $\operatorname{Spa}A$. The idea of the definition of $\mathcal{O}_A$ is the following. Let $U = R\left(\frac{T_1}{s_1}, \ldots, \frac{T_n}{s_n}\right) = \bigcap_{i=1}^n \{v \in \operatorname{Spa}A \mid v(t) \leqq v(s_i) \neq 0$ for all $t \in T_i\}$ be a rational subset of $\operatorname{Spa}A$. Of course, every element of $A^\triangleright$ should give an analytic function on U, and the functions given by $s_1, \ldots, s_n$ should be invertible. So every element of the ring $(A^\triangleright)_{s_1, \ldots, s_n} = A^\triangleright\left[\frac{1}{s_1}, \ldots, \frac{1}{s_n}\right]$ should induce a function on U. Furthermore, every function on U which can be "approximated" by functions from

(1.2) *Universal property of* $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$. (i) Let A be a f-adic ring, $s_1,\ldots,s_n$ elements of A and $T_1,\ldots,T_n$ finite subsets of A such that $T_i \cdot A$ is open in A for $i = 1,\ldots,n$. Then the topological ring $A\left(\frac{T_1}{s_n},\ldots,\frac{T_n}{s_n}\right)$ is f-adic, the canonical ring homomorphism $h: A \to A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ is continuous, $h(s_i) \in A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)^*$ and $\frac{h(t)}{h(s_i)} \in A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)^\circ$ for every $i = 1,\ldots,n, t \in T_i$, and if $f: A \to B$ is a continuous ring homomorphism from A to a f-adic ring B such as that $f(s_i) \in B^*$ and $\frac{f(t)}{f(s_i)} \in B^\circ$ for every $i = 1,\ldots,n, t \in T_i$ then there exists a unique continuous ring homomorphism $g: A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right) \to B$ with $f = g \circ h$.

(ii) Let A be an affinoid ring, $s_1,\ldots,s_n$ elements of $A^\triangleright$ and $T_1,\ldots,T_n$ finite subsets of $A^\triangleright$ such that $T_i \cdot A^\triangleright$ is open in $A^\triangleright$ for $i = 1,\ldots,n$. Then the canonical ring homomorphism $h: A \to A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ from A to the affinoid ring $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ is continuous, $h(s_i) \in \left(A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)^\triangleright\right)^*$ and $\frac{h(t)}{h(s_i)} \in A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)^+$ for every $i = 1,\ldots,n, t \in T_i$, and if $f: A \to B$ is a continuous ring homomorphism from A to an affinoid ring B such that $f(s_i) \in (B^\triangleright)^*$ and $\frac{f(t)}{f(s_i)} \in B^+$ for every $i = 1,\ldots,n, t \in T_i$, then there exists a unique continuous ring homomorphism $g: A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right) \to B$ with $f = g \circ h$.

As mentioned before, we want to define, for a rational subset $U = R\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ of the topological space Spa$A$, $\mathcal{O}_A(U) = A^\triangleright\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$. But we have to check that $\mathcal{O}_A(U)$ depends only on U and is independent of the choice of $s_1,\ldots,s_n, T_1,\ldots,T_n$. This is done in the next proposition using an idea of Tate in [T].

Proposition 1.3 *Let A be an affinoid ring and U a rational subset of* Spa$A$. *Then* (i) *There exists a continuous ring homomorphism $h: A \to F_A(U)$ from A to a complete affinoid ring $F_A(U)$ such that* $\mathrm{im}(\mathrm{Spa}(h)) \subseteq U$ *and whenever $f: A \to B$ is a continuous ring homomorphism from A to a complete affinoid ring B with* $\mathrm{im}(\mathrm{Spa}(f)) \subseteq U$ *then there is a unique continuous ring homomorphism $g: F_A(U) \to B$ with $f = g \circ h$.*
(ii) *Let $s_1,\ldots,s_n$ be elements of $A^\triangleright$ and $T_1,\ldots,T_n$ finite subsets of $A^\triangleright$ such that $T_i \cdot A^\triangleright$ is open in $A^\triangleright$ for $i = 1,\ldots,n$ and $U = R\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$. Then the canonical ring homomorphism $A \to A\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$ satisfies the property of* (i).

Proof. Let $U = R\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ be as in (ii) and let $h: A \to A\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$ be the canonical ring homomorphism. It follows immediately from the definition of $A\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$ that $\mathrm{im}(\mathrm{Spa}(h)) \subseteq U$. Let $f: A \to B$ be a continuous ring

homomorphism from A to a complete affinoid ring B such that $\mathrm{im}(\mathrm{Spa}(f)) \subseteq U$. Then $v(f(s_i)) \neq 0$ for every $i = 1,\ldots,n, v \in \mathrm{Spa}B$, and hence the subsequent Lemma 1.4 implies that $f(s_i) \in (B^\triangleright)^*$ for $i = 1,\ldots,n$. Furthermore, we have $v\left(\frac{f(t)}{f(s_i)}\right) \leqq 1$ for every $i = 1,\ldots,n, t \in T_i$, $v \in \mathrm{Spa}B$, and hence by [H1, 3.3.i] $\frac{f(t)}{f(s_i)} \in B^+$ for every $i = 1,\ldots,n, t \in T_i$. Now the universal property (1.2. ii) says that there exists a unique continuous homomorphism $g: A\left\langle \frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle \to B$ with $f = g \circ h$. □

Lemma 1.4 *Let A be a complete affinoid ring and $\mathfrak{m}$ a maximal ideal of $A^\triangleright$. Then there exists a point $v \in \mathrm{Spa}A$ with $\mathfrak{m} = \mathrm{supp}(v)$.*

Proof. We equip $B^\triangleright := A^\triangleright/\mathfrak{m}$ with the quotient topology. Let $B^+ \subseteq B^\triangleright$ be the integral closure of A^+ is $B^\triangleright$. Then $B := (B^\triangleright, B^+)$ is an affinoid ring, and the image of the natural mapping $\mathrm{Spa}B \to \mathrm{Spa}A$ is the set of points $v \in \mathrm{Spa}A$ with $\mathfrak{m} = \mathrm{supp}(v)$. Hence we have to show $\mathrm{Spa}B \neq \emptyset$. The set of units of $A^\triangleright$ is open, since $1 + (A^\triangleright)^{\circ\circ}$ is open and every element of $1 + (A^\triangleright)^{\circ\circ}$ is a unit of $A^\triangleright$. Hence $\mathfrak{m}$ is closed in $A^\triangleright$, i.e., $B^\triangleright$ is Hausdorff. Now [H1, 3.6.i] implies that $\mathrm{Spa}B \neq \emptyset$. □

The ring homomorphism $A \to F_A(U)$ in (1.3.i) is uniquely determined up to unique isomorphism. For the following we fix, for every affinoid ring A and every rational subset U of $\mathrm{Spa}A$, a ring homomorphism $h_{A,U}: A \to F_A(U)$ which has the property of (1.3.i). We note some properties of these ring homomorphisms.

Lemma 1.5 *Let A be an affinoid ring and U a rational subset of* $\mathrm{Spa}A$.

(i) *If V is a rational subset of* $\mathrm{Spa}A$ *with $V \subseteq U$ then there is a unique continuous ring homomorphism $g: F_A(U) \to F_A(V)$ with $h_{A,V} = g \circ h_{A,U}$.*
(ii) *The mapping $g := \mathrm{Spa}(h_{A,U}) : \mathrm{Spa}F_A(U) \to \mathrm{Spa}A$ is a homeomorphism from* $\mathrm{Spa}F_A(U)$ *onto U, and it induces an one-to-one correspondence between the rational subsets of* $\mathrm{Spa}F_A(U)$ *and the rational subsets of* $\mathrm{Spa}A$ *which are contained in U (i.e. if V is a rational subset of* $\mathrm{Spa}F_A(U)$ *then $g(V)$ is a rational subset of* $\mathrm{Spa}A$, *and if V is a rational subset of* $\mathrm{Spa}A$ *then $g^{-1}(V)$ is a rational subset of* $\mathrm{Spa}F_A(U)$).
(iii) *Put $B := F_A(U)$ and $g := \mathrm{Spa}(h_{A,U}): \mathrm{Spa}B \to \mathrm{Spa}A$. Let V be a rational subset of* $\mathrm{Spa}A$ *with $V \subseteq U$. Then there exists a unique continuous ring homomorphism $r: F_A(V) \to F_B(g^{-1}(V))$ such that the following diagram is commutative*

$$\begin{array}{ccc} F_B(g^{-1}(V)) & \xleftarrow{r} & F_A(V) \\ {\scriptstyle h_{B,\,g^{-1}(V)}}\uparrow & & \uparrow{\scriptstyle h_{A,V}} \\ B & \xleftarrow[h_{A,U}]{} & A \end{array}$$

Furthermore, r is an isomorphism (i.e., r has a continuous inverse).

Proof. (i) follows from (1.3.i).
(ii) Let s be an element of $A^\triangleright$ and T a finite subset of $A^\triangleright$ such that $T \cdot A^\triangleright$ is open in $A^\triangleright$ and $U = R\left(\frac{T}{s}\right)$. By (1.3.ii) we can assume that $h_{A,U}$ is the canoni-

polynomial ring $\mathbb{Z}[X_1, X_2, X_3]$ by X_1X_2, and let B be the subring of $A^\triangleright$ generated by $X_2, X_1X_2, X_1^{-1}X_2, (X_1^nX_2^{-n}X_3 \mid n \in \mathbb{N}), (X_1^{-n}X_2^{-n}X_3 \mid n \in \mathbb{N})$. We equip $A^\triangleright$ with the group topology such that $\{X_2^nB \mid n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0. Then $A^\triangleright$ is a Tate ring. Let A^+ be an arbitrary ring of integral elements of $A^\triangleright$, and put $A = (A^\triangleright, A^+)$. The topological space $\mathrm{Spa}A$ is covered by the open subsets $U = \{v \in \mathrm{Spa}A \mid v(X_1) \leqq 1\}$ and $V = \{v \in \mathrm{Spa}A \mid v(X_1) \geqq 1\}$. Let X be the element of $\mathcal{O}_A(\mathrm{Spa}A)$ given by X_3. Then $X|U = 0$ and $X|V = 0$, since X_3 lies in every neighbourhood of 0 in $A^\triangleright\left(\frac{1, X_1}{1}\right)$ and $A^\triangleright\left(\frac{1}{X_1}\right)$. But $X \neq 0$, since $X_3 \notin X_2B$ as a direct computation shows.

[Sch 18]
4.3.1.1, Thm. 3.12
p.17

2 Adic spaces

Let X be a topological space and F a presheaf of complete topological rings on X (i.e., F is a presheaf of rings on X and, for every open subset $U \subseteq X$, the ring $F(U)$ is equipped with a complete ring topology such that all restriction homomorphisms $F(V) \to F(W)$ are continuous). Then F is a sheaf of complete topological rings (in the sense of [EGA*, 0.3.1]) if and only if F is a sheaf of rings and for every open covering $(U_i)_{i \in I}$ of an open subset U of X the natural mapping $F(U) \to \prod_{i \in I} F(U_i)$ is a topological embedding, where $\prod_{i \in I} F(U_i)$ carries the product topology.

We will use the following category $\mathcal{V}$. The objects are the triples $X = (X, \mathcal{O}_X, (v_x \mid x \in X))$, where X is a topological space, $\mathcal{O}_X$ is a sheaf of complete topological rings on X and $v_x \in \mathrm{Spv}\mathcal{O}_{X,x}$ is a valuation of the stalk $\mathcal{O}_{X,x}$. ($\mathcal{O}_{X,x}$ denotes the inductive limit $\varinjlim_{x \in U} \mathcal{O}_X(U)$ in the category of rings.)

The morphisms $X \to Y$ are the pairs (f, φ), where $f: X \to Y$ is a continuous mapping and $\varphi: \mathcal{O}_Y \to f_*\mathcal{O}_Y$ is a morphism of sheaves of topological rings (i.e. φ is a morphism of sheaves of rings and, for every open subset $U \subseteq Y$, the mapping $\varphi_U: \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is continuous) such that, for every $x \in X$, the induced ring homomorphism $\varphi_x: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is compatible with the valuations v_x and $v_{f(x)}$ (i.e., $v_{f(x)} = \mathrm{Spv}(\varphi_x)(v_x)$).

In the last paragraph we constructed to every affinoid ring A a triple $(\mathrm{Spa}A, \mathcal{O}_A, (v_x | x \in \mathrm{Spa}A))$. We saw that, in general, $\mathcal{O}_A$ is not a sheaf. But if $\mathcal{O}_A$ is a sheaf of topological rings then $(\mathrm{Spa}A, \mathcal{O}_A, (v_x | x \in \mathrm{Spa}A))$ is an object of $\mathcal{V}$ which we call the adic space associated with A.

Definition. *An* **affinoid adic space** *is an object of $\mathcal{V}$ which is isomorphic to the adic space associated with an affinoid ring. An* **adic space** *is an object $(X, \mathcal{O}_X, (v_x | x \in X))$ of $\mathcal{V}$ which is locally an affinoid adic space, i.e., every $x \in X$ has an open neighbourhood $U \subseteq X$ such that $(U, \mathcal{O}_X \mid U, (v_x \mid x \in U))$ is an affinoid adic space. A morphism $X \to Y$ between adic spaces X, Y is a morphism in $\mathcal{V}$.*

The main aim of this paragraph is to show that for two important classes of affinoid rings A the presheaf $\mathcal{O}_A$ on $\mathrm{Spa}A$ is a sheaf of topological rings (Theorem 2.2). Furthermore, we will show that the morphisms $\mathrm{Spa}A \to \mathrm{Spa}B$ between the adic space $\mathrm{Spa}A$ and $\mathrm{Spa}B$ associated with complete affinoid rings

A and B correspond bijectively to the continuous ring homomorphism $B \to A$ (Proposition 2.1). With these two results we will see in Sect. 4 that the category of adic spaces is large enough that it contains both the category of locally noetherian formal schemes and the category of rigid analytic varieties.

We begin with some general remarks on adic spaces. Let $(X, \mathcal{O}_X, (v_x | x \in X))$ be an adic space. Then $\mathcal{O}_X^+$ denotes the subsheaf of $\mathcal{O}_X$ with $\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \mid v_x(f) \leqq 1 \text{ for all } x \in U\}$. $\mathcal{O}_X^+$ is a sheaf of rings. By (1.6.i, ii) the stalks $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}^+$ of the sheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are local rings. Furthermore, if $f: (X, \mathcal{O}_X, (v_x \mid x \in X)) \to (Y, \mathcal{O}_Y, (v_y \mid y \in Y))$ is a morphism of adic spaces then f induces morphisms of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y^+)$. (In (2.1) we will prove a converse of this.)

Let $(\mathrm{Spa}A, \mathcal{O}_A, (v_x \mid x \in \mathrm{Spa}A))$ be the adic space associated with an affinoid ring A. Then $(U, \mathcal{O}_A \mid U, (v_x \mid x \in U))$ is an affinoid adic space for every rational subset U of $\mathrm{Spa}A$ (by 1.5.ii, iii)). Hence open subspaces of adic spaces are adic spaces. If X is an affinoid adic space then $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ is an affinoid ring, and X is isomorphic to the adic space associated with $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ (by (1.5.ii, iii) and (1.6.iv)).

Let $A = (A^\triangleright, A^+)$ be an affinoid ring and $\hat{A} = ((A^\triangleright)^\wedge, (A^+)^\wedge)$ the completion of A. Then $\mathcal{O}_A$ is a sheaf of topological rings on $\mathrm{Spa}A$ if and only if $\mathcal{O}_{\hat{A}}$ is a sheaf of topological rings on $\mathrm{Spa}\hat{A}$, and the adic spaces associated with A and $\hat{A}$ are isomorphic. This follows from (1.5.ii, iii) with $U = \mathrm{Spa}A$ (note $F_A(\mathrm{Spa}A) \cong \hat{A}$).

Proposition 2.1 (i) *Let $X = \mathrm{Spa}B$ and $Y = \mathrm{Spa}A$ be the adic spaces associated with affinoid rings B and A. Then every continuous ring homomorphism $A \to B$ induces in a canonical way a morphism of adic spaces $X \to Y$. Thus we have a mapping from the set of continuous ring homomorphisms $A \to B$ to the set of morphisms of adic spaces $\mathrm{Spa}B \to \mathrm{Spa}A$. If B is complete then this mapping is bijective.*

(ii) *Let X be an adic space and $Y = \mathrm{Spa}A$ the adic space associated with an affinoid ring A. Then there is a natural one-to-one correspondence between the set of morphisms $X \to Y$ and the set of continuous ring homomorphisms $A \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$.*

(iii) *Let $X = (X, \mathcal{O}_X, (v_x \mid x \in X))$ and $Y = (Y, \mathcal{O}_Y, (v_y \mid y \in Y))$ be adic spaces, and let $g = (f, \varphi): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of topologically ringed spaces (i.e., (f, φ) is a morphism of ringed spaces and, for every open subset $U \subseteq Y$, the ring homomorphism $\varphi_U: \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is continuous). Then g is a morphism of adic spaces if and only if the following two conditions are satisfied.*

(a) *$g: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces.*

(b) *g induces a morphism of locally ringed space $(X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y^+)$ (i.e., the sheaf homomorphism $\varphi: \mathcal{O}_Y \to f_* \mathcal{O}_X$ satisfies $\varphi(\mathcal{O}_Y^+) \subseteq f_* \mathcal{O}_X^+$, and $(f, \psi): (X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y^+)$ is a morphism of locally ringed spaces, where $\psi: \mathcal{O}_Y^+ \to f_* \mathcal{O}_X^+$ is the restriction of φ).*

Proof. (i) Let $f: A \to B$ be a continuous ring homomorphism. We construct a morphism of adic spaces $s(f): X \to Y$. Let $g = \mathrm{Spa}(f): \mathrm{Spa}B \to \mathrm{Spa}A$ be the continuous mapping induced by f. By the universal property described

as well p.46
[Hub96]
adic space

Let A be a topological ring.
A is called adic if there exists an ideal I of A such that $\{I^n \mid n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of zero; I is called an ideal of definition of A.
A is called *f-adic* if there exists an open subring A_0 of A such that A_0 equipped with the subspace topology is adic and has a finitely generated ideal of definition; then every open subring of A whose subspace topology is adic is called a *ring of definition* of A; every ring of definition of A has a finitely generated ideal of definition.
A is called a *Tate ring* if A is f-adic and has a topologically nilpotent unit. For example, if we equip the localization B_s of a ring B with respect to an element $s \in B$ with the group topology such that $\{\mathrm{im}(s^n B \to B_s) \mid n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of zero then B_s is a Tate ring. Conversely, if A is a Tate ring, B a ring of definition of A and $s \in B$ a topologically nilpotent unit of A then $A = B_s$ and sB is an ideal of definition of B.

For a Tate ring A, let $A\langle X_1, \ldots, X_n\rangle$ denote the ring of all power series $\sum_{\nu \in \mathbb{N}_0^n} a_\nu X^\nu$ in n variables over the completion $\hat{A}$ of A such that $(a_\nu)_{\nu \in \mathbb{N}_0^n}$ is a zero sequence in A. We equip $A\langle X_1, \ldots, X_n\rangle$ with the group topology such that the sets $U\langle X\rangle := \{\sum_{\nu \in \mathbb{N}_0^n} a_\nu X^\nu \in A\langle X_1, \ldots, X_n\rangle \mid a_\nu \in U \text{ for every } \nu \in \mathbb{N}_0^n\}$ where U is a neighbourhood of 0 in A form a fundamental system of neighbourhoods of zero. Then $A\langle X_1, \ldots, X_n\rangle$ is a complete Tate ring. A is called *strongly noetherian* if, for every $n \in \mathbb{N}$, the ring $A\langle X_1, \ldots, X_n\rangle$ is noetherian.

For every topological ring A, let $A^{\circ\circ} \subseteq A$ denote the set of all topologically nilpotent elements of A and let $A^\circ \subseteq A$ denote the set of all power-bounded elements of A ($a \in A$ is called power-bounded if $\{a^n \mid n \in \mathbb{N}\}$ is bounded in A, i.e. for every neighbourhood U of 0 there is a neighbourhood V of 0 such that $\{a^n \mid n \in \mathbb{N}\} \cdot V \subseteq U$).

A *ring of integral elements* of a f-adic ring A is a subring B of A which is open in A, integrally closed in A and contained in the ring A° of power-bounded elements of A. An *affinoid ring* is a pair $A = (A^\flat, A^+)$ where A is a f-adic ring and A^+ is a ring of integral elements of $A^\flat$.

[Hub94], p.522

If not stated otherwise, throughout this book every f-adic ring A is assumed to satisfy one of the following conditions

(1.1.1) a) the completion $\hat{A}$ of A has a noetherian ring of definition over which $\hat{A}$ is finitely generated
b) A is a strongly noetherian Tate ring
c) the topology of A is discrete

(1.1.2) Let (V) denote the category of triples $(X, \mathcal{O}_X, (v_x \mid x \in X))$ where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings on X and, for every $x \in X$, v_x is an equivalence class of valuations of the stalk $\mathcal{O}_{X,x}$ (for our notations concerning valuations see below). For $a \in \mathcal{O}_{X,x}$, we write very often $|a(x)|$ instead of $v_x(a)$.
The morphisms $(X, \mathcal{O}_X, (v_x \mid x \in X)) \to (Y, \mathcal{O}_Y, (v_y \mid y \in Y))$ are the pairs (f, φ)

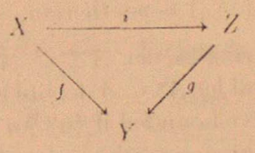
1 Adic spaces

Adic spaces are defined in [Hu], [Hu$_4$]. In §1 of this chapter we will recall the definition of adic spaces. In the further paragraphs of this chapter we will introduce some notions on adic spaces. (Some of these notions are already introduced in [Hu], [Hu$_4$].)

Since an affinoid adic space Spa$A$ is associated with a *pair* $A = (A^b, A^+)$ of rings, there are several notions of morphisms of finite type between adic spaces. These are introduced in §2. Furthermore, we will give in §2 a criterion for the existence of fibre products in the category of adic spaces.

In §3 we define proper morphisms of adic spaces and explain a valuative criterion for properness. This criterion plays an important role in our construction of compactifications of morphisms of adic spaces in Chapter 5.

In §7 we will show that every étale morphism between affinoid adic spaces $f: X \to Y$ has a factorization



where i is an open embedding and g is an étale algebraic morphism.

For establising the étale cohomology theory of rigid analytic varieties and adic spaces but also for applications of this theory, one has to study the étale toposes of subsets of adic spaces. Therefore in the last paragraph of this chapter we will introduce a category of subsets of adic spaces. An object of this category is a pair (X, S) where X is an adic space and S is a convex locally pro-constructible subset of (the locally spectral topological space) X. We call such a pair a pseudo-adic space. A morphism of pseudo-adic spaces $f: (X, S) \to (Y, T)$ is by definition a morphism of adic spaces $f: X \to Y$ with $f(S) \subseteq T$.

1.1 Definition of adic spaces

We fix some notations and recall the definition of adic spaces from [Hu], [Hu$_1$], [Hu$_3$].

In this book all rings are assumed to be commutative with unit element, and all complete topological groups are assumed to be hausdorff.

in (1.3.i) there exists, for every rational subset U of $\mathrm{Spa}B$ and every rational subset V of $\mathrm{Spa}A$ with $g(U) \subseteq V$, a unique continuous ring homomorphism $\varphi_{V,U}: F_A(V) \to F_B(U)$ such that the following diagram commutes

$$\begin{array}{ccc} F_B(U) & \xleftarrow{\varphi_{V,U}} & F_A(V) \\ {\scriptstyle h_{B,U}}\uparrow & & \uparrow{\scriptstyle h_{A,V}} \\ B & \xleftarrow[f]{} & A\,. \end{array}$$

These $\varphi_{V,U}$ induce a morphism of sheaves of topological rings $\varphi: \mathcal{O}_A \to g_*\mathcal{O}_B$. For every $x \in \mathrm{Spa}B$ the induced ring homomorphism $\mathcal{O}_{A,g(x)} \to \mathcal{O}_{B,x}$ is compatible with the valuations $v_{g(x)}$ and v_x. Hence $s(f) := (g, \varphi): X \to Y$ is a morphism of adic spaces.

Now assume that B is complete. Let $f_1, f_2: A \to B$ be continuous ring homomorphisms with $s(f_1) = s(f_2)$. Let $\varphi: \mathcal{O}_A(\mathrm{Spa}A) \to \mathcal{O}_B(\mathrm{Spa}B)$ be the ring homomorphism induced by $s(f_1) = s(f_2)$, and let $h_A: A^\triangleright \to \mathcal{O}_A(\mathrm{Spa}A)$ and $h_B: B^\triangleright \to \mathcal{O}_B(\mathrm{Spa}B)$ be the canonical ring homomorphisms. Then by the construction of $s(f_1)$ and $s(f_2)$ we have $h_B \circ f_1 = \varphi \circ h_A$ and $h_B \circ f_2 = \varphi \circ h_A$. Since B is complete and therefore h_B is an isomorphism, we obtain $f_1 = f_2$. Let $r: X \to Y$ be a morphism of adic spaces. We have to show that there exists a continuous ring homomorphism $f: A \to B$ with $r = s(f)$. Let $\varphi: F_A(\mathrm{Spa}A) = (\mathcal{O}_A(\mathrm{Spa}A), \mathcal{O}_A^+(\mathrm{Spa}A)) \to (\mathcal{O}_B(\mathrm{Spa}B), \mathcal{O}_B^+(\mathrm{Spa}B)) = F_B(\mathrm{Spa}B)$ be the ring homomorphism induced by r. Since $h_{B,\mathrm{Spa}B}$ is an isomorphism, we have a continuous ring homomorphism $f: A \to B$ such that the following diagram commutes

$$(1) \qquad \begin{array}{ccc} F_B(\mathrm{Spa}B) & \xleftarrow{\varphi} & F_A(\mathrm{Spa}A) \\ {\scriptstyle h_{B,\mathrm{Spa}B}}\uparrow & & \uparrow{\scriptstyle h_{A,\mathrm{Spa}A}} \\ B & \xleftarrow[f]{} & A\,. \end{array}$$

Since for every $x \in \mathrm{Spa}B$ the ring homomorphism $\mathcal{O}_{A,r(x)} \to \mathcal{O}_{B,x}$ induced by r is compatible with the valuations $v_{r(x)}$ and v_x, the commutativity of (1) shows $r(x) = \mathrm{Spa}(f)(x)$ for every $x \in \mathrm{Spa}B$. Let $U = R\left(\frac{T}{d}\right)$ be a rational subset of $\mathrm{Spa}A$. Put $V := r^{-1}(U) = s(f)^{-1}(U) \subset \mathrm{Spa}B$. Let $h_U: A^\triangleright \to \mathcal{O}_A(U)$ and $h_V: B^\triangleright \to \mathcal{O}_B(V)$ be the canonical ring homomorphisms, and let $\alpha: \mathcal{O}_A(U) \to \mathcal{O}_B(V)$ and $\beta: \mathcal{O}_A(U) \to \mathcal{O}_B(V)$ be the continuous ring homomorphisms induced by r and $s(f)$. Of course β makes the following diagram commutative

$$(2) \qquad \begin{array}{ccc} \mathcal{O}_B(V) & \longleftarrow & \mathcal{O}_A(U) \\ {\scriptstyle h_V}\uparrow & & \uparrow{\scriptstyle h_U} \\ B^\triangleright & \xleftarrow[f]{} & A^\triangleright. \end{array}$$

The commutativity of (1) implies that also α makes the diagram (2) commutative. Since d is invertible in $\mathcal{O}_A(U)$ and $A^\triangleright\left[\frac{1}{d}\right]$ is dense in $\mathcal{O}_A(U)$, we obtain $\alpha = \beta$. Hence $r = s(f)$.

(ii) To a morphism $f: X \to Y$ we assign the continuous ring homomorphism $A \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ which is the composition of the canonical ring homomorphism $A \to (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ and the ring homomorphism $(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y)) \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ induced by f. Thus we have a mapping from the set of morphisms $X \to Y$ to the set of continuous ring homomorphisms $A \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ which is bijective by (i).
(iii) Assume that (a) and (b) are satisfied. We have to show that for every $x \in X$ the mapping φ_x: $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is compatible with the valuations $v_{f(x)}$ and v_x, i.e., $v_{f(x)} = \mathrm{Spv}(\varphi_x)(v_x)$. By (1.6.i) the maximal ideals of $\mathcal{O}_{Y,f(x)}$ and $\mathcal{O}_{X,x}$ are the supports of $v_{f(x)}$ and v_x. Hence by (a) both $v_{f(x)}$ and $\mathrm{Spv}(\varphi_x)(v_x)$ have the support $\mathfrak{m}_{f(x)} \subseteq \mathcal{O}_{Y,f(x)}$. Let $h: \mathcal{O}_{Y,f(x)} \to k := \mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}$ be the canonical mapping, and let $A_{f(x)} \subset k$ and $A_x \subset k$ be the valuation rings to $v_{f(x)}$ and $\mathrm{Spv}(\varphi_x)(v_x)$. We have to show that $A_{f(x)} = A_x$. The rings $h^{-1}(A_{f(x)})$ and $h^{-1}(A_x)$ are local. By (1.6.ii) we have $h^{-1}(A_{f(x)}) = \mathcal{O}_{Y,f(x)}^+$ and $h^{-1}(A_x) = \varphi_x^{-1}(\mathcal{O}_{X,x}^+)$. So (b) implies that $h^{-1}(A_x)$ dominates $h^{-1}(A_{f(x)})$. Then A_x dominates $A_{f(x)}$, and hence $A_x = A_{f(x)}$. □

For every Tate ring A let $A\langle X_1, \ldots, X_n\rangle$ denote the subring $\{\Sigma a_\nu X^\nu \in \hat{A}[[X_1, \ldots, X_n]] \mid (a_\nu)_{\nu \in \mathbb{N}_0^n}$ is a zero sequence in $\hat{A}\}$ of $\hat{A}[[X_1, \ldots, X_n]]$. We equip $A\langle X_1, \ldots, X_n\rangle$ with the group topology such that $\{U\langle X\rangle \mid U$ neighbourhood of 0 in $\hat{A}\}$ is a fundamental system of neighbourhoods of 0 in $A\langle X_1, \ldots, X_n\rangle$, where $U\langle X\rangle = \{\Sigma a_\nu X^\nu \in A\langle X_1, \ldots, X_n\rangle \mid a_\nu \in U$ for all $\nu\}$. Then $A\langle X_1, \ldots, X_n\rangle$ is a complete Tate ring. We say that A is **strongly noetherian** if $A\langle X_1, \ldots, X_n\rangle$ is noetherian for every $n \in \mathbb{N}_0$. Now we can state the main result of this paragraph.

Theorem 2.2 *Let A be an affinoid ring such that $A^\triangleright$ has a noetherian ring of definition or $A^\triangleright$ is a strongly noetherian Tate ring. Then $\mathcal{O}_A$ is a sheaf of complete topological rings on* Spa$A$. *Furthermore, $H^i(U, \mathcal{O}_A) = 0$ for every $i \in \mathbb{N}$ and every rational subset U of* Spa$A$.

We will even prove a slight generalization of (2.2), namely a theorem analogous to (2.2) for $\mathcal{O}_A$-premodules. For that we need some preparations.

Let A be a topological ring and M a finitely generated A-module. We call the A-module topology on M such that every A-module homomorphism $M \to N$ from M to a topological A-module N is continuous the natural A-module topology of M. If $m_1, \ldots, m_s$ is a system of generators of M over A then the set of all sets $\{x_1 m_1 + \ldots + x_s m_s \mid x_1, \ldots, x_s \in U\}$ with U a neighbourhood of 0 in A is a fundamental system of neighbourhoods of 0 in the natural A-module topology of M. Now assume that A is f-adic with ring of definition A_0 and ideal of definition I of A_0. Let M_0 be a A_0-submodule of M, and equip M_0 with the I-adic topology. Then M_0 is an open subspace of M in the natural A-module topology of M if and only if M_0 is open and bounded in M. (A subset $M_0 \subseteq M$ is called bounded if for every neighbourhood U of 0 in M there exists a neighbourhood V of 0 in A with $\{a \cdot m \mid a \in V, m \in M_0\} \subseteq U$.)

A mapping $f: X \to Y$ between topological spaces X and Y is called strict if f is continuous and the restriction $X \to f(X)$ is open.

Lemma 2.3 *Let A be a f-adic ring which has a noetherian ring of a definition. Let M and N be finitely generated A-modules equipped with their natural A-module topology. Then.*

(i) *Every A-module homomorphism $f: M \to N$ is strict.*

(ii) *If A is complete then M is complete and every A-submodule of M is closed in M.*

(iii) *The natural mapping $M \otimes_A \hat{A} \to \hat{M}$ is an isomorphism of topological $\hat{A}$-modules if we equip $M \otimes_A \hat{A}$ with its natural $\hat{A}$-module topology.*

Proof. (i) and (ii) follow from analogous results for finitely generated modules over noetherian adic rings, and (iii) is a consequence of (ii). □

Lemma 2.4 (i) *Let A be a topological ring which has a zero sequence $(a_n \mid n \in \mathbb{N})$ with $a_n \in A^*$ for every $n \in \mathbb{N}$ (for example, A a Tate ring). Let M and N be topological A-modules which are complete and have countable fundamental systems of neighbourhoods of 0. Then every continuous surjective A-module homomorphism $M \to N$ is open.*

(ii) *Let A be a complete noetherian Tate ring, and let M and N be finitely generated A-modules equipped with their natural A-module topologies. Then M is complete and every A-module homomorphism $M \to N$ is strict.*

Proof. In order to prove (i) one can take over without any change the proof of Banach's open mapping theorem (cf. [B1, I.3.3.]). (ii) follows from (i) with the methods of [BGR, 3.7]. □

Let A be an affinoid ring and let M be a finitely generated $A^\triangleright$-module. For every rational subset W of Spa$A$ we equip the $\mathcal{O}_A(W)$-module $M \otimes_{A^\triangleright} \mathcal{O}_A(W)$ with the natural $\mathcal{O}_A(W)$-module topology. Then, for rational subsets V, W with $V \subseteq W$, the natural mapping $\rho_{V,W}: M \otimes_{A^\triangleright} \mathcal{O}_A(W) \to M \otimes_{A^\triangleright} \mathcal{O}_A(V)$ is continuous. For every open subset U of Spa$A$ we put

$$(M \otimes \mathcal{O}_A)(U) := \varprojlim_V M \otimes_{A^\triangleright} \mathcal{O}_A(V),$$

where the projective limit is taken over all rational subsets V of Spa$A$ contained in U and with respect to the transition mappings $\rho_{V,W}$. We equip $(M \otimes \mathcal{O}_A)(U)$ with the projective limit topology. Then $(M \otimes \mathcal{O}_A)(U)$ is a topological $\mathcal{O}_A(U)$-module, and for $V \subseteq U$ the canonical mapping $(M \otimes \mathcal{O}_A)(U) \to (M \otimes \mathcal{O}_A)(V)$ is continuous. So we have a topological $\mathcal{O}_A$-premodule $M \otimes \mathcal{O}_A$. Now we can generalize (2.2) to the following theorem

Theorem 2.5 *Let A be an affinoid ring such that $A^\triangleright$ has a noetherian ring of definition or $A^\triangleright$ is a strongly noetherian Tate ring, and let M be a finitely generated $A^\triangleright$-module. Then $M \otimes \mathcal{O}_A$ is a sheaf of complete topological groups on* Spa$A$, *and $H^i(U, M \otimes \mathcal{O}_A) = 0$ for every $i \in \mathbb{N}$ and every rational subset U of* Spa$A$.

Proof. We distinguish the cases that $A^\triangleright$ has a noetherian ring of definition and that $A^\triangleright$ is a strongly noetherian Tate ring. In the first case our proof is motivated by Raynaud's paper [R], and in the second case our proof is similar to the proof of Tate's acyclicity theorem in [BGR].

Case I. $A^\triangleright$ has a noetherian ring of definition.

First we describe a construction which generalizes the blowing up of ideals. Let $\varphi: C \to B$ be a ring homomorphism and let J be a C-submodule of B. Then we have the graduated C-algebra $\bigoplus_{n\in\mathbb{N}_0} J^n$, where $J^0 := \varphi(C)$ and, for $n \geqq 1, J^n$ is the subgroup of B generated by $j_1 \cdot \ldots \cdot j_n$ with $j_1, \ldots, j_n \in J$. We have a canonical morphism of schemes $g: \mathrm{Proj}(\bigoplus_{n\in\mathbb{N}_0} J^n) \to \mathrm{Spec}\ C$. Let I be the ideal of B generated by J. Then the inclusion $\bigoplus_{n\in\mathbb{N}_0} J^n \to \bigoplus_{n\in\mathbb{N}_0} I^n$ defines a morphism of schemes $s: \mathrm{Proj}(\bigoplus_{n\in\mathbb{N}_0} I^n) \to \mathrm{Proj}(\bigoplus_{n\in\mathbb{N}_0} J^n)$ such that the following diagram commutes

$$\begin{array}{ccc} X := \mathrm{Proj}(\bigoplus\limits_{n\in\mathbb{N}_0} I^n) & \longrightarrow & \mathrm{Spec}B \\ s \downarrow & & \downarrow \\ Y := \mathrm{Proj}(\bigoplus\limits_{n\in\mathbb{N}_0} J^n) & \xrightarrow[g]{} & \mathrm{Spec}C \end{array} \tag{I.1}$$

One can easily check

(I.2)

(i) s is affine.

(ii) Let F be a set of generators of the C-module J. Let S be a nonempty finite subset of $F \subseteq J^1$ and let $t \in J^{|S|}$ be the product of the elements of S. Put $U = D_+(t) = \{\mathfrak{p} \in \mathrm{Proj}(\bigoplus_{n\in\mathbb{N}_0} J^n) | t \notin \mathfrak{p}\} \subseteq Y$. Then one can consider $\mathcal{O}_Y(U)$ and $\mathcal{O}_X(s^{-1}(U))$ as subrings of the localization B_t, and we have $\mathcal{O}_Y(U) = C[\frac{f}{s} \mid f \in F, s \in S]$ and $\mathcal{O}_X(s^{-1}(U)) = B[\frac{f}{s} \mid f \in F, s \in S]$.

(iii) Let H be an ideal of C such that $\mathrm{Spec}(\varphi): \mathrm{Spec}B \to \mathrm{Spec}C$ induces an isomorphism $\mathrm{Spec}B - V(H \cdot B) \to \mathrm{Spec}C - V(H)$ and $V(I) \subseteq V(H \cdot B)$. Then g induces an isomorphism $Y - g^{-1}(V(H)) \to \mathrm{Spec}C - V(H)$.

The essential step in our proof of case I is the following point.

(I.3)

Let B be an affinoid ring, let $f_0, \ldots, f_n$ be elements of B° with $B^\circ = f_0 B^\circ + \ldots + f_n B^\circ$, and let P be a finitely generated B°-module. We assume that B° has a noetherian ring of definition. Put $\mathcal{F} := P \otimes \mathcal{O}_B, Z := \mathrm{Spa}B$, and $U_i := R\left(\frac{f_0, \ldots, f_n}{f_i}\right) \subseteq Z$ for $i = 0, \ldots, n$. Then the augmented Cech complex to $\mathcal{F}$ and the covering $\{U_0, \ldots, U_n\}$ of Z

$$(*) \qquad 0 \to \mathcal{F}(Z) \to \prod_{i_0} \mathcal{F}(U_{i_0}) \to \prod_{(i_0, i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1}) \to \ldots$$

is exact. Furthermore, if we equip all components of $(*)$ with their natural topologies then all differentials of $(*)$ are strict.

Proof of (I.3) Let $\tilde{P}$ be the quasi-coherent sheaf on $\mathrm{Spec}B^\circ$ defined by P. We consider the augmented Cech complex to $\tilde{P}$ and the covering $\{D(f_0), \ldots, D(f_n)\}$ of $X = \mathrm{Spec}B^\circ$

$$(**) \qquad 0 \to \tilde{P}(X) \to \prod_{i_0} \tilde{P}(D(f_{i_0})) \to \prod_{(i_0, i_1)} \tilde{P}(D(f_{i_0}) \cap D(f_{i_1})) \to \ldots$$

Put $F = \{f_0, \ldots, f_n\}$. We equip $\tilde{P}(X) = P$ with the natural B°-module topology. For every $(f_{i_0}, \ldots, f_{i_k}) \in F^{k+1}$, the underlying ring of the topo-

logical ring $B^\triangleright\left(\frac{F}{f_{i_0}},\ldots,\frac{F}{f_{i_k}}\right)$ is the localization of $B^\triangleright$ by $f_{i_0}\cdot\ldots\cdot f_{i_k}$. So $\tilde{P}(D(f_{i_0})\cap\ldots\cap D(f_{i_k})) = P\otimes_{B^\triangleright} B^\triangleright\left(\frac{F}{f_{i_0}},\ldots,\frac{F}{f_{i_k}}\right)$, and hence $\tilde{P}(D(f_{i_0})\cap\ldots\cap D(f_{i_k}))$ is a finitely generated $B^\triangleright\left(\frac{F}{f_{i_0}},\ldots,\frac{F}{f_{i_k}}\right)$-module. We equip $\tilde{P}(D(f_{i_0})\cap\ldots\cap D(f_{i_k}))$ with the natural $B^\triangleright\left(\frac{F}{f_{i_0}},\ldots,\frac{F}{f_{i_k}}\right)$- module topology, and $\prod_{(i_0,\ldots,i_k)\in F^{k+1}}\tilde{P}(D(f_{i_0})\cap\ldots\cap D(f_{i_k}))$ with the product topology. Then every differential of $(**)$ is continuous. Let $(**)^\wedge$ be the completion of the complex $(**)$. By (2.3. iii), $(**)^\wedge$ is the complex $(*)$. The complex $(**)$ is exact. If all differentials of $(**)$ are strict then by [B2, III.2.12 Lemma 2] all differentials of $(**)^\wedge$ are strict and $(**)^\wedge$ is exact. So it is sufficient to show that all differentials of $(**)$ are strict.

Let C be a noetherian ring of definition of $B^\triangleright$, and let J be the C-submodule of $B^\triangleright$ generated by F. Since $B^\triangleright = f_0B^\triangleright + \ldots + f_nB^\triangleright$, we have by (I.1) a commutative diagram

$$\begin{array}{ccc} & & \operatorname{Spec}B^\triangleright = X \\ & \overset{s}{\swarrow} & \downarrow h \\ Y := \operatorname{Proj}\left(\bigoplus_{n\in\mathbb{N}_0} J^n\right) & \xrightarrow[g]{} & \operatorname{Spec}C\,. \end{array}$$

Let N be a finitely generated C-submodule of P which generates the $B^\triangleright$-module P, and let $\tilde{N}$ be the coherent sheaf on $\operatorname{Spec}C$ defined by N. The inclusion $N\to P$ induces a morphism of sheaves $\tau:\tilde{N}\to h_*(\tilde{P})$. The composition of $g^*(\tau):g^*(\tilde{N})\to g^*h_*(\tilde{P}) = g^*g_*s_*(\tilde{P})$ with the adjunction morphism $g^*g_*s_*(\tilde{P})\to s_*(\tilde{P})$ gives a morphism of sheaves $\sigma:g^*(\tilde{N})\to s_*(\tilde{P})$. The image $\mathcal{G}$ of σ is a coherent sheaf on Y, since $g^*(\tilde{N})$ is coherent and $s_*(\tilde{P})$ is quasi-coherent.

Let I be an ideal of definition of C. For every $i\in\mathbb{N}$ let $K_i^\bullet$ be the augmented Cech complex to the sheaf $I^i\mathcal{G}$ and the covering $\{D_+(f_0),\ldots,D_+(f_n)\}$ of Y. Since $D(f_k) = s^{-1}(D_+(f_k))$ for $k=0,\ldots,n$, we can identify the complex $(**)$ with the augmented Cech complex $K_0^\bullet$ to the sheaf $s_*(\tilde{P})$ and the covering $\{D_+(f_0),\ldots,D_+(f_n)\}$ of Y. Since $s_*(\tilde{P})\supseteq\mathcal{G}\supseteq I\mathcal{G}\supseteq I^2\mathcal{G}\supseteq\ldots$, we have a sequence of subcomplexes $K_0^\bullet\supseteq K_1^\bullet\supseteq K_2^\bullet\supseteq\ldots$. For every $i\in\mathbb{N}_0$ let $d_i^p:K_i^p\to K_i^{p+1}$ be the differentials of the complex $K_i^\bullet$. Then we have

(I.3.1)

(i) Given $p\in\mathbb{Z}$ and $u\in\mathbb{N}$ then there exists a $v\in\mathbb{N}_0$ with $\operatorname{im}(d_u^p)\supseteq\ker(d_v^{p+1})$.

(ii) For every $p\in\mathbb{Z}$, the set $\{K_i^p \mid i\in\mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 in K_0^p.

Proof. (i) The assertion is trivial if $p<0$. So we fix a $p\in\mathbb{N}_0$. For every $x\in I^r$ and $y\in H^{p+1}(Y,I^u\mathcal{G})$ let $x\odot y\in H^{p+1}(Y,I^{u+r}\mathcal{G})$ be the image of y under the mapping $H^{p+1}(Y,I^u\mathcal{G})\to H^{p+1}(Y,I^{u+r}\mathcal{G})$ induced by the x-multiplication $I^u\mathcal{G}\to I^{u+r}\mathcal{G}$. Then by [EGA, III. 3.3.2] there exists a $k\in\mathbb{N}$ such that

(1) $H^{p+1}(Y,I^{u+r}\mathcal{G}) = I^r\odot H^{p+1}(Y,I^u\mathcal{G})$ for every $u\geqq k$ and $r\geqq 0$.

It is enough to proof (I.3.1. i) for $u \geqq k$. We fix a $u \geqq k$. By [H1, 3.7], $\mathrm{Spec}B^\triangleright - V(I \cdot B^\triangleright) \to \mathrm{Spec}C - V(I)$ is an isomorphism. Hence (I.2. iii) implies that $Y - g^{-1}(V(I)) \to \mathrm{Spec}C - V(I)$ is an isomorphism. The sheaf $H^{p+1}(Y, I^u\mathscr{G})^\sim$ on $\mathrm{Spec}\,C$ associated with the finitely generated C-module $H^{p+1}(Y, I^u\mathscr{G})$ is equal to the higher direct image sheaf $R^{p+1}g_*(I^u\mathscr{G})$. Hence the restriction of $H^{p+1}(Y, I^u\mathscr{G})^\sim$ to $\mathrm{Spec}\ C - V(I)$ vanishes. Then according to [EGA*, I.6.8.4] there exists a $t \in \mathbb{N}$ with

(2) $I^t H^{p+1}(Y, I^u\mathscr{G}) = 0$.

We show $\mathrm{im}(d_u^p) \supseteq \ker(d_{u+t}^{p+1})$. Notice that $H^n(K_i^\bullet) = H^n(Y, I^i\mathscr{G})$ for every $n, i \in \mathbb{N}$. Let $a \in \ker(d_{u+t}^{p+1})$ be given, and let $\bar{a} \in H^{p+1}(Y, I^{u+t}\mathscr{G})$ be the cohomology class represented by a. By (1) there exist $x_1, \ldots, x_w \in I^t$ and $\bar{y}_1, \ldots, \bar{y}_w \in H^{p+1}(Y, I^u\mathscr{G})$ with $\bar{a} = x_1 \odot \bar{y}_1 + \ldots + x_w \odot \bar{y}_w$. Let $y_1, \ldots, y_w$ be elements of $\ker(d_u^{p+1})$ which represent the cohomology classes $\bar{y}_1, \ldots, \bar{y}_w$. Then $a - (x_1 y_1 + \ldots + x_w y_w) \in \mathrm{im}(d_{u+t}^p) \subseteq \mathrm{im}(d_u^p)$. By (2) we have $x_j y_j \in \mathrm{im}(d_u^p)$ for $j = 1, \ldots, w$. Hence $a \in \mathrm{im}(d_u^p)$.

(ii) Let S be a non empty subset of F. Then $C[\frac{f}{s} \mid s \in S,\ f \in F]$ is a ring of definition of the f-adic ring $B^\triangleright(\frac{F}{s} \mid s \in S)$, and $L := I \cdot C[\frac{f}{s} \mid s \in S,\ f \in F]$ is an ideal of definition of $C[\frac{f}{s} \mid s \in S,\ f \in F]$. Let G be the image of $N \otimes_C C[\frac{f}{s} \mid s \in S, f \in F]$ in $P \otimes_{B^\triangleright} B^\triangleright(\frac{F}{s} \mid s \in S)$. Then $\{L^i \cdot G \mid i \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 in the natural $B^\triangleright(\frac{F}{s} \mid s \in S)$-module topology of $P \otimes_{B^\triangleright} B^\triangleright(\frac{F}{s} \mid s \in S)$. By (I.2.ii) we have $\mathscr{O}_Y(\bigcap_{s\in S} D_+(s)) = \mathscr{O}_Y(D_+(\prod_{s\in S} s)) = C[\frac{f}{s} \mid s \in S, f \in F]$. Hence in case $p \geqq 0$ the assertion of (I.3.1.ii) follows immediately from our construction of the sheaf $\mathscr{G}$. Now we assume $p = -1$. By construction of $\mathscr{G}$ we have $I^i N \subseteq \Gamma(Y, I^i\mathscr{G})$ for every $i \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed. We have to show that there exists a $i \in \mathbb{N}$ with $\Gamma(Y, I^i\mathscr{G}) \subseteq I^k N$. By [EGA, III.3.3.2] there exists a $r \in \mathbb{N}$ such that $\Gamma(Y, I^{r+s}\mathscr{G}) = I^s\Gamma(Y, I^r\mathscr{G})$ for every $s \in \mathbb{N}$. Since $\Gamma(Y, I^r\mathscr{G})$ is a finitely generated C-submodule of P and $I^k N$ is an open C-submodule of P, there exists a $t \in \mathbb{N}$ with $I^t\Gamma(Y, I^r\mathscr{G}) \subseteq I^k N$. Hence $\Gamma(Y, I^{r+t}\mathscr{G}) \subseteq I^k N$.

Now we can show that all differentials of the complex $(**) = K_0^\bullet$ are strict. Let $p \in \mathbb{Z}$ be given. Applying (I.3.1.ii) to K_0^{p+1}, we see that $\mathrm{im}(d_0^p) \cap \ker(d_i^{p+1}) = \mathrm{im}(d_0^p) \cap K_i^{p+1}$ is a neighbourhood of 0 in $\mathrm{im}(d_0^p)$ for every $i \in \mathbb{N}$. Now (I.3.1.i, ii) shows that $d_0^p : K_0^p \to \mathrm{im}(d_0^p)$ is open. This concludes the proof of (I.3).

By (1.5) we have

(I.4)
Let U be a rational subset of $\mathrm{Spa}A$. Put $B = F_A(U)$ and $P = M \otimes_{A^\triangleright} B^\triangleright$. Then there exists a homeomorphism $f : \mathrm{Spa}B \to U$ and a morphism of presheaves $\varphi : (M \otimes \mathscr{O}_A) \mid U \to f_*(P \otimes \mathscr{O}_B)$ such that f induces a bijection between the set of the rational subsets of $\mathrm{Spa}B$ and the set of the rational subsets of $\mathrm{Spa}A$ contained in U, and, for every rational subset V of $\mathrm{Spa}A$ with $V \subseteq U$, the mapping $\varphi(V) : (M \otimes \mathscr{O}_A)(V) \to (P \otimes \mathscr{O}_B)(f^{-1}(V))$ is an isomorphism of topological groups.

Note that, for every rational subset U of Spa$A$, $F_A(U)^\triangleright$ is complete and has a noetherian ring of definition. Hence, applying (I.3), (I.4) and the subsequent Lemma 2.6, we obtain

(I.5)

Let $(V_j)_{j\in J}$ be an open covering of a rational subset U of Spa$A$. Then there exist rational subsets $U_1,\dots,U_n$ of Spa$A$ such that $U=\bigcup_{i=1}^n U_i$, every U_i is contained in some V_j, and the augmented Cech complex to $\mathscr{F}:=M\otimes\mathscr{O}_A$ and the covering $(U_i \mid i=1,\dots,n)$ of U

$$0\to\mathscr{F}(U)\to\prod_{i_0}\mathscr{F}(U_{i_0})\to\prod_{(i_0,\,i_1)}\mathscr{F}(U_{i_0}\cap U_{i_1})\to\dots$$

is exact and has strict differentials.

Now we are ready to finish the proof of (2.5) in our first case. We have to show that $M\otimes\mathscr{O}_A$ is a sheaf of complete topological groups and that, for every rational subset U of Spa$A$ and every $i\in\mathbb{N}$, $H^i(U,M\otimes\mathscr{O}_A)=0$. By (2.3.ii), $(M\otimes\mathscr{O})(U)$ is complete for every rational subset U of Spa$A$, and then $(M\otimes\mathscr{O})(U)$ is complete for every open subset U of Spa$A$ (by definition of $M\otimes\mathscr{O}_A$). So we have a presheaf of complete topological groups. In order to show that $M\otimes\mathscr{O}_A$ is a sheaf of complete topological groups it is enough to show that, for every rational subset U of Spa$A$ and every covering $(U_i)_{i\in I}$ of U by rational subsets of Spa$A$, the sequence

$$(+)\quad 0\longrightarrow(M\otimes\mathscr{O}_A)(U)\xrightarrow{d}\prod_i(M\otimes\mathscr{O}_A)(U_i)\longrightarrow\prod_{(i,\,j)}(M\otimes\mathscr{O}_A)(U_i\cap U_j)$$

is exact and d is strict [EGA*, 0.3.2.2]. Then it is easy to see that it is even sufficient to show that, for every rational subset U of Spa$A$ and every covering $(V_j)_{j\in J}$ of U by rational subsets of Spa$A$, there exists a refinement $(U_i)_{i\in I}$ of $(V_j)_{j\in J}$ by rational subsets U_i of Spa$A$ such that $(+)$ is exact and d is strict. But this is covered by (I.5). Likewise by (I.5) we have $\check{H}^i(U,M\otimes\mathscr{O}_A)=0$ for every $i\in\mathbb{N}$ and every rational subset U of Spa$A$. Then [G, 3.8 Corollary 4] implies $H^i(U,M\otimes\mathscr{O}_A)=0$ for every $i\in\mathbb{N}$ and every rational subset U of Spa$A$.

Case II. $A^\triangleright$ is a strongly noetherian Tate ring.

In the following two points (II.1) and (II.2) let B be an affinoid ring such that $B^\triangleright$ is a strongly noetherian Tate ring.

(II.1)

(i) Let $B^\triangleright\langle X,X^{-1}\rangle$ be the ring of all formal series $\sum_{n\in\mathbb{Z}} b_nX^n$ such that $b_n\in(B^\triangleright)^\wedge$ for every $n\in\mathbb{Z}$ and, for every neighbourhood U of 0 in $(B^\triangleright)^\wedge$, $b_n\notin U$ for only finitely many $n\in\mathbb{Z}$. Then the $B^\triangleright$-algebras $B^\triangleright\langle X,X^{-1}\rangle$ and $B^\triangleright\langle X,Y\rangle/(1-XY)$ are canonically isomorphic.

(ii) For $i=1,\dots,m$ let $T_i=\{f_{i1},\dots,f_{in(i)},g_i\}$ be a finite subset of $B^\triangleright$ with $B^\triangleright=T_i\cdot B^\triangleright$. Put $C=B^\triangleright\langle X_{ij} \mid i=1,\dots,m,j=1,\dots,n(i)\rangle$ and let I be the ideal of C generated by $\{f_{ij}-g_iX_{ij} \mid i=1,\dots,m,j=1,\dots,n(i)\}$. Then the topological $B^\triangleright$-algebras $B^\triangleright\left\langle\frac{T_1}{g_1},\dots,\frac{T_m}{g_m}\right\rangle$ and C/I are canonically isomorphic.

(iii) For every rational subset U of $\mathrm{Spa}B$, the Tate ring $\mathcal{O}_B(U)$ is strongly noetherian.

(iv) For every rational subset U of $\mathrm{Spa}B$, $\mathcal{O}_B(U)$ is flat over $\mathcal{O}_B(\mathrm{Spa}B)$.

Proof. (i) We equip $B^\triangleright\langle X, X^{-1}\rangle$ with the group topology such that the sets $\{\sum_{n\in\mathbb{Z}} b_n X^n \in B^\triangleright\langle X, X^{-1}\rangle \mid b_n \in U \text{ for every } n \in \mathbb{Z}\}$ (U neighbourhood of 0 in $(B^\triangleright)^\wedge$) form a fundamental system of neighbourhoods of 0 in $B^\triangleright\langle X, X^{-1}\rangle$. Then $B^\triangleright\langle X, X^{-1}\rangle$ is a complete Tate ring. By (2.4.ii), the ideal $(1 - XY)$ is closed in $B^\triangleright\langle X, Y\rangle$ and hence $B^\triangleright\langle X, Y\rangle/(1 - XY)$ is a complete Tate ring. Both ring homomorphisms $B^\triangleright \to (B^\triangleright\langle X, X^{-1}\rangle; X)$ and $B^\triangleright \to (B^\triangleright\langle X, Y\rangle/(1 - XY); X)$ are universal with respect to continuous ring homomorphisms from $B^\triangleright$ to complete Tate rings with a distinguished unit u such that u and u^{-1} are power-bounded (cf. (3.3.i)).

(ii) By (2.4.ii), I is closed in C, and hence C/I is a complete f-adic ring. Every g_i is a unit in C/I, since $B^\triangleright = T_i \cdot B^\triangleright$. Now it is easily seen that both ring homomorphisms $B^\triangleright \to B^\triangleright\left\langle \frac{T_1}{g_1}, \ldots, \frac{T_m}{g_m}\right\rangle$ and $B^\triangleright \to C/I$ satisfy the same universal property, cf. (1.2) and (3.3.i).

(iii) follows from (3.4.i, ii).

(iv) We proceed as in the proof of [FP, III.7.10]. First we show that it is enough to prove the assertion only for some special rational subsets U. For that we use (1.5). Namely let $f_1, \ldots, f_n, g$ be elements of $B^\triangleright$ with $U = \{x \in \mathrm{Spa}B \mid x(f_i) \leqq x(g) \neq 0 \text{ of } i = 1, \ldots, n\}$. Since U is quasi-compact, there exists by [H1, 3.11] a unit s of $B^\triangleright$ with $x(s) \leqq x(g)$ for every $x \in U$. Put $Y_1 := \{x \in \mathrm{Spa}B \mid 1 \leqq x\left(\frac{g}{s}\right)\}$. Then $g \mid Y_1$ is a unit of $\mathcal{O}_B(Y_1)$, and so we can define inductively rational subsets $Y_2, \ldots, Y_{n+1}$ of $Y_0 := \mathrm{Spa}B$ with $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_{n+1} = U$ by $Y_k = \{x \in Y_{k-1} \mid x\left(\frac{f_{k-1}}{g}\right) \leqq 1\}$ $(k = 2, \ldots, n+1)$. Then the restriction $h: \mathcal{O}_B(\mathrm{Spa}B) \to \mathcal{O}_B(U)$ factorizes into $h = h_n \circ \ldots \circ h_0$, where $h_i: \mathcal{O}_B(Y_i) \to \mathcal{O}_B(Y_{i+1})$ is the restriction. Using (2.4.ii) and (ii) one can prove with the ideas of [FP, III.7.8, 7.9] that, for every complete affinoid ring D such that $D^\triangleright$ is noetherian and Tate and for every $f \in D^\triangleright$, the restrictions $\mathcal{O}_D(\mathrm{Spa}D) \to \mathcal{O}_D(U_1)$ and $\mathcal{O}_D(\mathrm{Spa}D) \to \mathcal{O}_D(U_2)$ are flat, where $U_1 = \{x \in \mathrm{Spa}D \mid x(f) \geqq 1\}$ and $U_2 = \{x \in \mathrm{Spa}D \mid x(f) \leqq 1\}$. Hence by (iii), all ring homomorphisms h_i are flat.

(II.2)

Let f be an element of $B^\triangleright$, and put $U_1 = \{x \in \mathrm{Spa}B \mid x(f) \leqq 1\}$ and $U_2 = \{x \in \mathrm{Spa}B \mid x(f) \geqq 1\}$. Then the augmented Cech complex to $\mathcal{O}_B$ and the covering $\{U_1, U_2\}$ of $\mathrm{Spa}B$

$$0 \longrightarrow \mathcal{O}_B(\mathrm{Spa}\ B) \xrightarrow{\varepsilon} \mathcal{O}_B(U_1) \times \mathcal{O}_B(U_2) \xrightarrow{\delta} \mathcal{O}_B(U_1 \cap U_2) \longrightarrow 0$$

is exact.

Proof. By (II.1.iv) the ring homomorphism ε is flat. With (1.4) we obtain that ε is faithfully flat and hence injective. By (II.1.i, ii) we have $\mathcal{O}_B(U_1) = B^\triangleright\langle X\rangle/(f - X), \mathcal{O}_B(U_2) = B^\triangleright\langle Y\rangle/(1 - fY)$ and $\mathcal{O}_B(U_1 \cap U_2) = B^\triangleright\langle X, Y\rangle/(f - X, 1 - fY) = B^\triangleright\langle X, Y\rangle/(f - X, 1 - XY) = B^\triangleright\langle X, X^{-1}\rangle/(f - X)$. Now an easy computation shows that δ is surjective and $\mathrm{im}(\varepsilon) = \ker(\delta)$ (cf. [BGR, 8.2.3]).

(II.3)
Let $(V_j)_{j\in J}$ be an open covering of a rational subset U of Spa$A$. Then there exist rational subsets $U_1,\ldots,U_n$ of Spa$A$ such that $U=\bigcup_{i=1}^n U_i$, every U_i is contained in some V_j, and the augmented Cech complex to $M\otimes\mathcal{O}_A$ and the covering $(U_i|i=1,\ldots,n)$ of U is exact.

Proof. By (II.1.iv) it is sufficient to prove (II.3) for $M=A^\triangleright$ (use the alternating Cech complexes). (I.4) holds also in our case II. Hence by (II.1.iii) we can assume that $A^\triangleright$ is complete and $U=\mathrm{Spa}A$. Then by the subsequent lemma we can assume $(V_j)_{j\in J}=\left(R\left(\frac{f_0,\ldots,f_m}{f_i}\right)|i=0,\ldots,m\right)$ with some $f_0,\ldots,f_m\in A^\triangleright$ with $A^\triangleright=f_0A^\triangleright+\ldots+f_mA^\triangleright$. Now using (II.2) one can prove (II.3) with the reasonings of [BGR, 8.2.2].

(II.3) implies that $M\otimes\mathcal{O}_A$ is a sheaf of groups and $H^i(U,M\otimes\mathcal{O}_A)=0$ for every $i\in\mathbb{N}$ and every rational subset U of Spa$A$. By (2.4.ii) and (II.1.iii), $(M\otimes\mathcal{O}_A)(U)$ is complete for every rational subset U of Spa$A$. Then $(M\otimes\mathcal{O}_A)(U)$ is complete for every open subset U of Spa$A$. It remains to show that $M\otimes\mathcal{O}_A$ is a sheaf of topological groups. Let $(U_i)_{i\in I}$ be an open covering of an open subset U of Spa$A$. We have to show that $\varepsilon\colon(M\otimes\mathcal{O}_A)(U)\to\prod_{i\in I}(M\otimes\mathcal{O}_A)(U_i)$ is strict. By [EGA*, 0.3.2.2] we may assume that U and all U_i are rational, and since U is quasi-compact we may assume that I is finite. $\mathrm{im}(\varepsilon)$ is closed in $\prod_{i\in I}(M\otimes\mathcal{O}_A)(U_i)$, since $\mathrm{im}(\varepsilon)$ is the kernel of $\prod_{i\in I}(M\otimes\mathcal{O}_A)(U_i)\to\prod_{i,\,j\in I}(M\otimes\mathcal{O}_A)(U_i\cap U_j)$. Now (2.4.i) implies that ε is strict. □

Lemma 2.6 *Let A be a complete affinoid ring, and let $(V_j)_{j\in J}$ be an open covering of* Spa$A$. *Then there exist $f_0,\ldots,f_n\in A^\triangleright$ such that $A^\triangleright=f_0A^\triangleright+\ldots+f_nA^\triangleright$ and, for every $i\in\{0,\ldots,n\}$, the rational subset $R\left(\frac{f_0,\ldots,f_n}{f_i}\right)$ is contained in some V_j.*

Proof. Let $x\in\mathrm{Spa}A\subseteq\mathrm{Spv}A^\triangleright$. Let $c\Gamma_x$ be the characteristic subgroup of Γ_x as defined in [H1]. Let y be the valuation $x\mid c\Gamma_x$ of $A^\triangleright$. Then $\Gamma_y=c\Gamma_y$ and $y\in\mathrm{Spa}A$. We choose a V_j with $y\in V_j$. Since $\Gamma_y=c\Gamma_y$, there exists by the proof of [H1, 2.6.ii] an element $s\in A^\triangleright$ and a finite subset T of $A^\triangleright$ such that $1\in T$ and $y\in R\left(\frac{T}{s}\right)\subseteq V_j$. We have $x\in R\left(\frac{T}{s}\right)$, since x is a generalization of y. Using that Spa$A$ is quasi-compact, we obtain that the there exist elements $s_1,\ldots,s_n\in A^\triangleright$ and finite subsets $T_1,\ldots,T_n$ of $A^\triangleright$ such that $1\in T_i$ for every $i=1,\ldots,n$ and $\left(R\left(\frac{T_i}{s_i}\right)|i=1,\ldots,n\right)$ is a covering of Spa$A$ refining $(V_j)_{j\in J}$. We may assume $s_i\in T_i$ for $i=1,\ldots,n$.

Now we can follow the proof of [FP, III.2.5]. Put $T=\{t_1\cdot\ldots\cdot t_n|t_i\in T_i$ for $i=1,\ldots,n\}$ and $S=\{t_1\cdot\ldots\cdot t_n|t_i\in T_i$ for $i=1,\ldots,n$ and $t_i=s_i$ for at least one $i\in\{1,\ldots,n\}\}$. We note the following trivial properties

(1) For every $t_1\in T_1,\ldots,t_n\in T_n$ we have $R\left(\frac{T}{t}\right)=\bigcap_{i=1}^n R\left(\frac{T_i}{t_i}\right)$ with $t:=t_1\cdot\ldots\cdot t_n$.

(2) For every $i\in\{1,\ldots,n\}$, $\mathrm{Spa}A=\bigcup_{t\in T_i}R\left(\frac{T_i}{t}\right)$.

By (1), $R\left(\frac{T}{s}\right)$ is contained in some $R\left(\frac{T_i}{s_i}\right)$ for every $s \in S$. (1) and (2) imply

(3) $\mathrm{Spa}A = \bigcup_{s\in S} R\left(\frac{T}{s}\right)$.

From (1.4) and (3) we obtain $A^\triangleright = S \cdot A^\triangleright$. Furthermore, (3) implies $R\left(\frac{T}{s}\right) = R\left(\frac{S}{s}\right)$ for every $s \in S$. Hence every $R\left(\frac{S}{s}\right)$ is contained in some V_j. □

3 Adic morphisms, morphisms of finite type and fibre products

The following definition turns out to be very useful.

Definition. *A point x of an adic space X is called* **analytic** *if there exists an open neighbourhood U of x in X such that $\mathcal{O}_X(U)$ has a topologically nilpotent unit. We put $X_{\mathrm{a}} = \{x \in X | x \text{ is analytic}\}$ and $X_{\mathrm{na}} = X - X_{\mathrm{a}}$. If every point of X is analytic then X is called analytic.*

Remark 3.1 If $X = \mathrm{Spa}A$ is the adic space to an affinoid ring A then $x \in X$ is analytic if and only if $\mathrm{supp}(x) \in \mathrm{Spec}A^\triangleright$ is not open in $A^\triangleright$. (So the above definition coincides with the definition of an analytic point in [H1].)

Proof. Let x be a point of $\mathrm{Spa}A$ such that $\mathrm{supp}(x)$ is open in $A^\triangleright$, and let U be an open neighbourhood of x in $\mathrm{Spa}A$. We have to show that $\mathcal{O}_X(U)$ has no topologically nilpotent unit. Let V be a rational subset of $\mathrm{Spa}A$ with $x \in V \subseteq U$. Put $\mathfrak{p} = \{f \in \mathcal{O}_X(V) \mid v_x(f) = 0\}$. Then $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_X(V)$ with $\mathfrak{p} \cap A^\triangleright = \mathrm{supp}(x)$. Since $\mathrm{supp}(x)$ is open in $A^\triangleright$, it follows immediately from the definition of $\mathcal{O}_X(V)$ that $\mathfrak{p}$ is open $\mathcal{O}_X(V)$. Hence every topologically nilpotent element of $\mathcal{O}_X(V)$ lies in $\mathfrak{p}$. But since $\mathfrak{p}$ contains no unit of $\mathcal{O}_X(V)$, $\mathcal{O}_X(V)$ has no topologically nilpotent unit. Now let x be a point of $\mathrm{Spa}A$ such that $\mathrm{supp}(x)$ is not open in $A^\triangleright$. Then there exists a topologically nilpotent element s of $A^\triangleright$ with $x(s) \neq 0$. Put $U = \{y \in \mathrm{Spa}A \mid y(s) \neq 0\}$. The U is an open neighbourhood of x in $\mathrm{Spa}A$, and the image of s in $\mathcal{O}_X(U)$ is a topologically nilpotent unit of $\mathcal{O}_X(U)$. □

In [H1] we called a ring homomorphism $f: A \to B$ between f-adic rings adic if there exist rings of definition A_0, B_0 of A, B and an ideal of definition I of A_0 such that $f(A_0) \subseteq B_0$ and $f(I) \cdot B_0$ is an ideal of definition of B_0.

Definition. *A morphism $f: X \to Y$ between adic space is called* **adic** *if for every $x \in X$ there exist an open affinoid neighbourhood U of x in X and an open affinoid subspace V of Y such that $f(U) \subseteq V$ and the ring homomorphism between f-adic rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ induced by f is adic.*

(3.1) and [H1, 3.8] imply

Proposition 3.2 *Let $f: X \to Y$ be a morphism of adic spaces. Then*

(i) *If f is adic then, for all open affinoid subspaces U and V of X and Y with $f(U) \subseteq V$, the ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is adic.*

(ii) *f is adic if and only if $f(X_{\mathrm{a}}) \subseteq Y_{\mathrm{a}}$.*

factors uniquely over $W(R^+/\varpi)$, by obvious congruences; the induced map $W(R^+/\varpi) \to R^+/\varpi^m$ must be a ring homomorphism. Now the composite

$$W(R^{\flat +}) \to W(R^+/\varpi) \to R^+/\varpi^m ,$$

where the first map is given by the m-th component map $R^{\flat +} = \varprojlim_{x \mapsto x^p} R^+/\varpi \to R^+/\varpi$, is a ring map, which we claim is equal to θ modulo ϖ^m. This is a direct verification from the definitions.

For surjectivity of θ, choose a pseudouniformizer $\varpi^\flat$ of $R^\flat$ such that the p-th power of $\varpi = (\varpi^\flat)^\sharp$ divides p. We know that $R^{\flat +} \to R^+/\varpi$ is surjective, which shows that θ mod $[\varpi^\flat]$ is surjective. As everything is $[\varpi^\flat]$-adically complete, this implies that θ is surjective.

For part (ii), we claim first that there exists $f \in \varpi^\flat R^{\flat +}$ such that $f^\sharp \equiv p \pmod{p\varpi R^+}$. Indeed, consider $\alpha = p/\varpi \in R^+$. There exists $\beta \in R^{\flat +}$ such that $\beta^\sharp \equiv \alpha \pmod{pR^+}$. Then $(\varpi^\flat \beta)^\sharp = \varpi\alpha \equiv p \pmod{p\varpi R^+}$, and we can take $f = \varpi^\flat \beta$.

Thus we can write $p = f^\sharp + p\varpi^\sharp \sum_{n \geq 0} r_n^\sharp p^n$, with $r_n \in R^{\flat +}$. We can now define $\xi = p - [f] - [\varpi^\flat] \sum_{n \geq 0} [r_n] p^{n+1}$, which is of the desired form, and which lies in the kernel of θ. Finally we need to show that ξ generates $\ker(\theta)$. For this, note that θ induces a surjective map $f : W(R^{\flat +})/\xi \to R^+$. It is enough to show that f is an isomorphism modulo $[\varpi^\flat]$, because $W(R^{\flat +})/\xi$ is $[\varpi^\flat]$-torsion free and $[\varpi^\flat]$-adically complete. But

$$W(R^{\flat +})/(\xi, [\varpi^\flat]) = W(R^{\flat +})/(p, [\varpi^\flat]) = R^{\flat +}/\varpi^\flat = R^+/\varpi ,$$

as desired. □

It is now straightforward to check the following theorem.

Theorem 3.17 ([KL15], [Fon13]). *There is an equivalence of categories between:*

(i) *Pairs (S, S^+) of a perfectoid Tate ring S and an open and integrally closed subring $S^+ \subset S^\circ$, and*

(ii) *Triples $(R, R^+, \mathcal{J})$, where R is a perfectoid Tate ring of characteristic p, $R^+ \subset R^\circ$ is an open and integrally closed subring, and $\mathcal{J} \subset W(R^+)$ is an ideal which is primitive of degree 1.*

The functors are given by $(S, S^+) \mapsto (S^\flat, S^{\flat +}, \ker \theta)$ and $(R, R^+, \mathcal{J}) \mapsto (W(R^+)[[\varpi]^{-1}]/\mathcal{J}, W(R^+)/\mathcal{J})$, where $\varpi \in R$ is any pseudouniformizer. □

Now we can define perfectoid spaces.

Theorem 3.18 ([Sch12, Thm. 6.3],[KL15, Thm. 3.6.14]). *Let R be a perfectoid Tate ring with an open and integrally closed subring $R^+ \subset R^\circ$, and let $X = \mathrm{Spa}(R, R^+)$. Then $\mathcal{O}_X$ is a sheaf, and for all rational subsets $U \subset X = \mathrm{Spa}(R, R^+)$, $\mathcal{O}_X(U)$ is again perfectoid.*

If $(R^\flat, R^{\flat +})$ is the tilt of (R, R^+) and $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat +})$, then recall that by Theorem 3.12, $|X| \cong |X^\flat|$, identifying rational subsets. Moreover, for any rational subset $U \subset X$ with image $U^\flat \subset X^\flat$, $\mathcal{O}_X(U)$ is perfectoid with tilt $\mathcal{O}_{X^\flat}(U^\flat)$.

Definition 3.19. *A* perfectoid space *is an adic space covered by open subspaces which are isomorphic to $\mathrm{Spa}(R, R^+)$, where R is a perfectoid Tate ring, and $R^+ \subset R^\circ$ is an open and integrally closed subring.*

The tilting process glues to give a functor $X \mapsto X^\flat$. Theorem 3.13 globalizes to the following result.

bijection between the sets of open and integrally closed subrings of $R^{\flat\circ}$ and R°. This correspondence can be made more explicit:

Lemma 3.11. *The set of open and integrally closed subrings $R^+ \subset R^\circ$ is in bijection with the set of open and integrally closed subrings $R^{\flat+} \subset R^{\flat\circ}$, via $R^{\flat+} = \varprojlim_{x \mapsto x^p} R^+$. Also, $R^{\flat+}/\varpi^\flat = R^+/\varpi$.* □

The following two theorems belong to a pattern of "tilting equivalence".

Theorem 3.12 ([Sch12], [KL15])**.** *Let R be a perfectoid Tate ring with an open and integrally closed subring $R^+ \subset R$, with tilt $(R^\flat, R^{\flat+})$.*

The map sending $x \in \mathrm{Spa}(R, R^+)$ to $x^\flat \in \mathrm{Spa}(R^\flat, R^{\flat+})$ defined by $|f(x^\flat)| = |f^\sharp(x)|$ defines a homeomorphism $\mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(R^\flat, R^{\flat+})$. A subset $U \subset \mathrm{Spa}(R, R^+)$ is rational if and only if its image in $\mathrm{Spa}(R^\flat, R^{\flat+})$ is rational.

Theorem 3.13 ([Sch12])**.** *Let R be a perfectoid Tate ring with tilt $R^\flat$. Then there is an equivalence of categories between perfectoid R-algebras and perfectoid $R^\flat$-algebras, via $S \mapsto S^\flat$.*

Proof. In [Sch12], these are only proved over a perfectoid field, but the proof works in general. □

Let us describe the inverse functor, along the lines of Fontaine's Bourbaki talk, [Fon13]. In fact we will answer a more general question. Given a perfectoid Tate ring R in characteristic p, what are all the untilts $R^\sharp$ of R?

Lemma 3.14. *Let R be a perfectoid Tate ring with a ring of integral elements $R^+ \subset R$, and let $(R^\flat, R^{\flat+})$ be its tilt.*

(i) *There is a canonical surjective ring homomorphism*

$$\theta : W(R^{\flat+}) \to R^+$$
$$\sum_{n \geq 0} [r_n] p^n \mapsto \sum_{n \geq 0} r_n^\sharp p^n$$

(ii) *The kernel of θ is generated by a nonzero-divisor ξ of the form $\xi = p + [\varpi]\alpha$, where $\varpi \in R^{\flat+}$ is a pseudo-uniformizer, and $\alpha \in W(R^{\flat+})$.*

Related results are discussed in [Fon13], [FF11] and [KL15, Theorem 3.6.5].

Definition 3.15. *An ideal $I \subset W(R^{\flat+})$ is primitive of degree 1 if I is generated by an element of the form $\xi = p + [\varpi^\flat]\alpha$, with $\varpi^\flat \in R^{\flat+}$ a pseudo-uniformizer and $\alpha \in W(R^{\flat+})$.*

An element ξ of this form is necessarily a non-zero-divisor:

Lemma 3.16. *Any ξ of the form $\xi = p + [\varpi^\flat]\alpha$, with $\varpi^\flat \in R^{\flat+}$ a pseudo-uniformizer and $\alpha \in W(R^+)$, is a non-zero-divisor.*

Proof. Assume that $\xi \sum_{n \geq 0} [c_n] p^n = 0$. Modulo $[\varpi^\flat]$, this reads $\sum_{n \geq 0} [c_n] p^{n+1} \equiv 0 \pmod{[\varpi^\flat]}$, meaning that all $c_n \equiv 0 \pmod{\varpi^\flat}$. We can then divide all c_n by $\varpi^\flat$, and induct. □

Proof of Lemma 3.14. For part (i), we first check that θ is a ring map. It is enough to check modulo ϖ^m for any $m \geq 1$. For this, we use that the m-th ghost map

$$W(R^+) \to R^+/\varpi^m \ : \ (x_0, x_1, \ldots) \mapsto \sum_{n=0}^{m} x_n^{p^{m-n}} p^n$$

The rational subsets of Spa$A$ are quasi-compact and form a basis of the topology of Spa$A$. With every rational subset U of Spa$A$ we can associate a complete topological ring $\mathcal{O}_A(U)$. Namely, if U has a representation $(*)$ then $\mathcal{O}_A(U) := A^\flat(\frac{f_1,\dots,f_n}{g}) := (A^\flat(\frac{f_1,\dots,f_n}{g}))^\wedge$ where $A^\flat(\frac{f_1,\dots,f_n}{g})$ is the localization $(A^\flat)_g$ equipped with the I-adic topology such that if B is a ring of definition of $A^\flat$ and I is an ideal of definition of B then $B[\frac{f_1}{g},\dots,\frac{f_n}{g}]$ is a ring of definition of $(A^\flat)_g$ and $I \cdot B[\frac{f_1}{g},\dots,\frac{f_n}{g}]$ is an ideal of definition of $B[\frac{f_1}{g},\dots,\frac{f_n}{g}]$. Then $\mathcal{O}_A(U)$ depends only on U, and if V is a rational subset of Spa$A$ with $V \subseteq U$ then there is a natural continuous ring homomorphism $\mathcal{O}_A(U) \to \mathcal{O}_A(V)$. For every open subset W of Spa$A$, we put

$$\mathcal{O}_A(W) := \varprojlim_{\substack{U \subseteq W \\ U \text{ rational}}} \mathcal{O}_A(U).$$

Then $\mathcal{O}_A$ is a sheaf of complete topological rings on Spa$A$. For every $x \in \mathrm{Spa}A$, the valuation x of $A^\flat$ extends in a natural way to a valuation v_x of the stalk $\mathcal{O}_{A,x}$. Thus we obtain an object $(\mathrm{Spa}A, \mathcal{O}_A, (v_x \mid x \in \mathrm{Spa}A))$ of the category (V) which we call the *spectrum of* A or the *adic space associated with* A. For every $x \in X$, the stalk $\mathcal{O}_{A,x}$ is a local ring, and the support of v_x is the maximal ideal of $\mathcal{O}_{A,x}$. The completion $A := ((A^\flat)^\wedge, (A^+)^\wedge)$ of A is naturally isomorphic to $(\mathcal{O}_A(\mathrm{Spa}A), \mathcal{O}_A^+(\mathrm{Spa}A))$.

An *affinoid adic space* is an object of (V) which is isomorphic to the adic space associated with an affinoid ring. An *adic space* is an object of (V) which is locally an affinoid adic space. The morphisms between adic spaces are the morphisms in (V). The morphisms of adic spaces $X \to \mathrm{Spa}A$ correspond bijectively to the continuous ring homomorphisms $A \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$.

In (1.1.11) resp. (1.1.12) we will explain the relation between the category of adic spaces and the category of rigid analytic varieties resp. relative rigid spaces ([BGR], [BL]).

A point x of an adic space X is called *analytic* if there is an open neighbourhood U of x in X such that the topological ring $\mathcal{O}_X(U)$ has a topologically nilpotent unit. Thus a point $x \in X$ is analytic if and only if x has an open affinoid neighbourhood U in X such that $\mathcal{O}_X(U)$ is a Tate ring. X_a denotes the open subset of analytic points of X. If $X = X_a$ then X is called *analytic*. (If $X = \mathrm{Spa}A$ then $x \in X$ is analytic if and only if the support $\mathrm{supp}(x) \subseteq A^\flat$ is not open in $A^\flat$).

Definition 1.1.3. A *non-archimedean field* is a topological field k whose topology is induced by a valuation of rank 1 of k (but the topology of k is not necessarily complete).

In rigid analytic geometry all occuring valuation rings have rank 1. Working with analytic adic spaces the appropriate class of valuation rings is the class of microbial valuation rings as defined in the following definition. (For an analytic point x of an adic space X, the valuation v_x is microbial).

where $f: X \to Y$ is a continuous mapping and $\varphi: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of topological rings such that, for every $x \in X$, the induced ring homomorphism $\varphi_x: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is compatible with $v_{f(x)}$ and v_x (i.e., $v_{f(x)} = v_x \circ \varphi_x$). For every object $(X, \mathcal{O}_X, (v_x \mid x \in X))$ of (V), $\mathcal{O}_X^+$ denotes the subsheaf of $\mathcal{O}_X$ with $\mathcal{O}_X^+(U) := \{s \in \mathcal{O}_X(U) \mid v_x(s) \leq 1 \text{ for every } x \in U\}$ $(U \subseteq X \text{ open})$.

With every affinoid ring A we can associate an object of (V) as follows.

First we recall some notations on valuations. A *valuation* of a ring B is a mapping $x: B \to \Gamma \cup \{0\}$ where Γ is a totally ordered, abelian, multiplicative group such that, if one extends the ordering and the multiplication of Γ to $\Gamma \cup \{0\}$ by $0 \leq \gamma$ and $0 \cdot \gamma = \gamma \cdot 0 = 0$ for every $\gamma \in \Gamma \cup \{0\}$, then the following conditions are satisfied: $x(a \cdot b) = x(a) \cdot x(b)$ and $x(a + b) \leq \max\{x(a), x(b)\}$ for $a, b \in B$, and $x(0) = 0$, $x(1) = 1$ ([B, VI.3.1]).
Instead of $x(a)$ $(\in \Gamma \cup \{0\})$ we very often write $|a(x)|$.
The set $x^{-1}(0)$ is a prime ideal of B which is called the *support* of x and is denoted by $\operatorname{supp}(x)$.
Two valuations $x: B \to \Gamma \cup \{0\}$ and $y: B \to \Delta \cup \{0\}$ of B are called *equivalent* if there exists an isomorphism of ordered monoids $c: \Gamma' \cup \{0\} \to \Delta' \cup \{0\}$ with $y = c \circ x$ where Γ' and Δ' are the subgroups of Γ and Δ generated by $\operatorname{im}(x) - \{0\}$ and $\operatorname{im}(y) - \{0\}$. The equivalence classes of valuations of B correspond bijectively to the pairs $(\mathfrak{p}, V)$ where $\mathfrak{p}$ is a prime ideal of B and V is a valuation ring of the residue class field $\operatorname{qf}(B/\mathfrak{p})$.
Let $\operatorname{Spv} B$ denote the set of all equivalence classes of valuations of B. We equip $\operatorname{Spv} B$ with the topology which is generated by the sets $\{x \in \operatorname{Spv} B \mid |a(x)| \leq |b(x)| \neq 0\}$ $(a, b \in B)$. Then $\operatorname{Spv} B$ is a spectral space. (For our notations on spectral spaces, cf. (1.1.13)).
If B carries a ring topology then we call a valuation $x: B \to \Gamma \cup \{0\}$ *continuous* if, for every $\gamma \in \Gamma'$, the set $x^{-1}(\{\delta \in \Gamma \cup \{0\} \mid \delta < \gamma\})$ is open in B (or, equivalently, if we equip the residue class field $\operatorname{qf}(B/x^{-1}(0))$ with the topology induced by the valuation ring $V \subseteq \operatorname{qf}(B/x^{-1}(0))$ corresponding to x ([B, VI.5.1]) then the natural mapping $B \to \operatorname{qf}(B/x^{-1}(0))$ is continuous).
In this book we need only elementary properties of valuations. For the readers convenience, some of these are represented in (1.1.14).

Now let $A = (A^b, A^+)$ be an affinoid ring. We put

$$\operatorname{Spa} A := \{x \in \operatorname{Spv} A^b \mid x \text{ is continuous and } |a(x)| \leq 1 \text{ for every } a \in A^+\} \subseteq \operatorname{Spv} A^b$$

and equip $\operatorname{Spa} A$ with the subspace topology of $\operatorname{Spv} A^b$. Then $\operatorname{Spa} A$ is a spectral space. A subset U of $\operatorname{Spa} A$ is called *rational* if there exist $f_1, \ldots, f_n, g \in A^b$ such that the ideal $f_1 A^b + \cdots + f_n A^b$ is open in A^b and

$$(*) \qquad U = R\left(\frac{f_1, \ldots, f_n}{g}\right) := \{x \in \operatorname{Spa} A \mid |f_i(x)| \leq |g(x)| \neq 0 \text{ for } i = 1, \ldots, n\}$$

Let $f: X \to Y$ be a morphism of adic spaces. Then, for every $x \in X$ and $y := f(x) \in Y$, the induced morphism $k(y) \to k(x)$ is continuous and hence gives a morphism of adic spaces $\mathrm{Spa}\kappa(x) \to \mathrm{Spa}\kappa(y)$.

(1.1.8) Let X be an adic space and A an affinoid field. The morphisms of adic spaces $\mathrm{Spa}A \to X$ correspond bijectively to the pairs (y, φ) where y is a point of X and $\varphi: k(y) \to (A^\circ)^\wedge$ is a continuous ring homomorphism with $k(y)^+ = \varphi^{-1}((A^+)^\wedge)$. In particular, if $A = \kappa(x)$ for some $x \in X$ then the pair $(x, k(x) \to k(x)^\wedge)$ gives a morphism $\mathrm{Spa}\kappa(x) \to X$.

(1.1.9) A point $y \in X$ of a topological space X is called a generalization of a point $x \in X$ (or x is called a specialization of y) if y is contained in every neighbourhood of x in X. In [Hu$_1$, §2] we defined, for every ring B, certain generalizations (resp. specializations) of the topological space $\mathrm{Spv}B$ which we call secondary (resp. primary) generalizations (resp. secondary (resp. primary) specializations). Let X be an adic space. A point $y \in X$ is called a *secondary (resp. primary) generalization* of a point $x \in X$ (or x is called a *secondary (resp. primary) specialization* of y) if there is an open affinoid subspace $U = \mathrm{Spa}A$ of X such that $x, y \in U$ and y is a secondary (resp. primary) generalization of x in $\mathrm{Spv}A^\circ$. (Then every open affinoid neighbourhood $U = \mathrm{Spa}A$ of x in X contains y and y is a secondary (resp. primary) generalization of x in $\mathrm{Spv}A^\circ$). For every $x \in X$, the natural morphism $\mathrm{Spa}\kappa(x) \to X$ ((1.1.8)) gives a homeomorphism from $\mathrm{Spa}\kappa(x)$ to the set of all secondary generalizations of x in X. It is important that, for every analytic point x of X, all generalizations of x in X are secondary generalizations ([Hu$_1$, §3]).

For points x, y of a topological space X, we write $x \succ y$ if y is a specialization of x. We call a point x of X maximal if x is maximal with respect to the relation $\succ$.

In the following lemma we describe some special properties of the specializations of analytic adic spaces which are frequently used in this paper.

Lemma 1.1.10. *Let $f: X \to Y$ be a morphism of analytic adic spaces.*
i) *For every $x \in X$, the set $G(x)$ of all generalizations of x in X forms a chain, i.e. if $y, z \in G(x)$ then $y \succ z$ or $z \succ y$.*
ii) *A point $x \in X$ is a maximal point of X if and only if the valuation v_x has rank 1.*
iii) *Let x, y be points of X with $x \succ y$. Then the natural ring homomorphism $j: \mathcal{O}_{X,y} \to \mathcal{O}_{X,x}$ is local and flat. In particular, j is injective. Let $i: k(y) \to k(x)$ be the ring homomorphism induced by j. Then i is a homeomorphism onto its image, $i(k(y))$ is dense in $k(x)$, and $i(k(y)^+) = k(x)^+$.*
iv) *If x is a maximal point of X then $f(x)$ is a maximal point of Y.*
v) *Let x be a point of X, $y := f(x)$. Let $G(x)$ (resp. $G(y)$) be the set of all generalizations of x (resp. y) in X (resp. Y). Then $f(G(x)) = G(y)$.*
vi) *Let x_0, x_1 be points of X with $x_0 \succ x_1$. Then $f(\{x \in X \mid x_0 \succ x \succ x_1\}) = \{y \in Y \mid f(x_0) \succ y \succ f(x_1)\}$.*

Definition 1.1.4. A valuation ring A is called *microbial* if it satisfies the following equivalent conditions

a) $\mathrm{qf}(A)$ equipped with the valuation topology of A is a non-archimedean field ((1.1.3)).

b) $\mathrm{qf}(A)$ equipped with the valuation topology of A is a Tate ring.

c) $\mathrm{qf}(A)$ equipped with the valuation topology of A has a topologically nilpotent unit.

d) A equipped with the valuation topology of A is non-discrete and adic.

e) A has a prime ideal of height 1.

For example, every valuation ring A with $0 < \dim A < \infty$ is microbial.

If A is a microbial valuation ring and $\mathfrak{p} \subseteq A$ the prime ideal of height 1 then $\mathfrak{p} = A^{\circ\circ} = \mathrm{qf}(A)^{\circ\circ}$ and, for every $a \in \mathfrak{p} - \{0\}$, aA is an ideal of definition of A and $\mathrm{qf}(A) = A_a$.

Definition 1.1.5. An *affinoid field* is an affinoid ring A which satisfies the following conditions

a) A^b is a non-archimedean field or A^b is a field equipped with the discrete topology.

b) A^+ is a valuation ring of A^b.

If A^b is a non-archimedean field then A is called an *analytic affinoid field*. A is called *(separably) algebraically closed* if A^b is (separably) algebraically closed.

If A is an analytic affinoid field then the valuation ring A^+ is microbial and induces the topology of A^b.

(1.1.6) Let A be an affinoid field. The points of $\mathrm{Spa}A$ are the valuation rings B of A^b with $A^+ \subseteq B \subseteq (A^b)^\circ$. (If the topology of A^b is discrete then $(A^b)^\circ = A^b$ and if the topology of A^b is not discrete then $(A^b)^\circ$ is a valuation ring of rank 1 of A^b). Hence for every $x, y \in \mathrm{Spa}A$, x is a specialization of y or y is a specialization of x (cf. (1.1.9) below). $(A^b)^\circ$ is the maximal (= generic) point of $\mathrm{Spa}A$, and A^+ is the minimal (= closed) point of $\mathrm{Spa}A$. For every non-empty open subset U of $\mathrm{Spa}A$, we have $\mathcal{O}_A(U) = (A^b)^\wedge$.

Definition 1.1.7. Let x be a point of an adic space X.

i) $k(x)$ denotes the residue class field of the local ring $\mathcal{O}_{X,x}$, and $k(x)^+$ denotes the valuation ring of $k(x)$ which corresponds to the valuation e_x of $\mathcal{O}_{X,x}$. (If x is an analytic point of X then $k(x)^+$ is microbial).

ii) We equip $k(x)$ with a topology: If x is analytic then we equip $k(x)$ with the valuation topology induced by $k(x)^+$, and if x is non-analytic then we equip $k(x)$ with the discrete topology.

iii) $\kappa(x)$ denotes the affinoid field $(k(x), k(x)^+)$.

Corollary 3.20. *Let X be a perfectoid space with tilt $X^\flat$. Then the functor $X' \mapsto X'^\flat$ from perfectoid spaces over X to perfectoid spaces over $X^\flat$ is an equivalence of categories.*

Finally, let us recall some almost mathematics. Let R be a perfectoid Tate ring.

Definition 3.21. *An R°-module M is* almost zero *if $\varpi M = 0$ for all pseudo-uniformizers ϖ. Equivalently, if ϖ is a fixed pseudo-uniformizer admitting p-power roots, then M is almost zero if and only if $\varpi^{1/p^n} M = 0$ for all n.*

A similar definition applies to R^+-modules; in fact, the condition on M only uses the R^+-module structure.

Example 3.22. (i) If K is a perfectoid field, then the residue field $k = \mathcal{O}_K/\mathfrak{m}_K$ is almost zero as $\mathcal{O}_K$-module, where $\mathfrak{m}_K \subset \mathcal{O}_K$ is the maximal ideal. Conversely, any almost zero module over $\mathcal{O}_K$ is a k-vector space, and thus a direct sum of copies of k.

(ii) If R is perfectoid and $R^+ \subset R^\circ$ is any ring of integral elements, then R°/R^+ is almost zero. Indeed, if ϖ is a pseudo-uniformizer, and $x \in R^\circ$, then ϖx is topologically nilpotent. Since R^+ is open, there exists n with $(\varpi x)^n \in R^+$, so that $\varpi x \in R^+$ by integral closedness.

Extensions of almost zero modules are almost zero. Thus the category of almost zero modules is a thick Serre subcategory of the category of all modules, and one can take the quotient.

Definition 3.23. *The category of* almost R°-modules, *written $R^{\circ a}$-mod, is the quotient of the category of R°-modules by the subcategory of almost zero modules.*

One can also define R^{+a}-mod, and the natural forgetful map $R^{\circ a}$-mod $\to R^{+a}$-mod is an equivalence.

Theorem 3.24 ([Sch12], [KL15])**.** *Let R be a perfectoid Tate ring with an open and integrally closed subring $R^+ \subset R^\circ$, and let $X = \mathrm{Spa}(R, R^+)$. Then the R^+-module $H^i(X, \mathcal{O}_X^+)$ is almost zero for $i > 0$, and $H^0(X, \mathcal{O}_X^+) = R^+$.*

4. Set-theoretic bounds

In this paper, we will have to deal with many "large" constructions. However, we want to avoid set-theoretic issues, and in particular want to avoid the use of universes. The cleanest way to handle this seems to be to fix a certain cut-off cardinal κ.

Lemma 4.1. *There is an uncountable cardinal κ with the following properties.*

(i) *For all cardinals $\lambda < \kappa$, one has $2^\lambda < \kappa$, i.e. κ is a strong limit cardinal.*

(ii) *For all countable sequences $\lambda_1, \lambda_2, \ldots < \kappa$ of cardinals less than κ, the supremum of all λ_i is less than κ, i.e. the cofinality of κ is larger than ω.*

(iii) *For all cardinals $\lambda < \kappa$, there is a strong limit cardinal $\kappa_\lambda < \kappa$ such that the cofinality of κ_λ is larger than λ.*

Note that in (iii), the set of such κ_λ is automatically cofinal (as for any $\lambda \leq \lambda' < \kappa$, one has $\kappa_{\lambda'} \geq \lambda'$, and $\kappa_{\lambda'}$ has cofinality larger than λ). Intuitively, cutting off at κ means the following. We are always allowed to take power sets, and countable unions. Moreover, we are allowed to do any construction that takes less than κ many steps, as long as the cardinality of the objects we are manipulating is uniformly bounded above by a cardinal strictly less than κ. Indeed, if for example

we want to take a union of $\lambda < \kappa$ many objects of size less than $\kappa' < \kappa$, then we can assume that $\kappa' = \kappa_\lambda$ is of cofinality larger than λ, in which case the union is still of size less than $\kappa_\lambda < \kappa$. Moreover, if at each of the λ intermediate steps we are doing some auxiliary large construction that however respects sets that are less than κ_λ, then this is still admissible.

Proof. First, we show that for any cardinal λ, one can find a strong limit cardinal κ such that the cofinality of κ is larger than λ. For this, one starts with the countable cardinal $\beth_0 = \aleph_0$ and then defines by transfinite induction $\beth_\mu$ for any ordinal μ as the power set of $\beth_{\mu^-}$ if μ is the successor of μ^-, or the union of all previous $\beth_{\mu'}$ if μ is a limit ordinal. Then $\beth_\mu$ is a strong limit cardinal with cofinality larger than λ, where μ is any ordinal with cofinality larger than λ (for example, the smallest ordinal of cardinality larger than λ).

Now build a sequence of cardinals $\kappa(0), \kappa(1), \dots$ indexed by ordinals μ, where $\kappa(0)$ is a strong limit cardinal of cofinality $> \omega$, and for a successor ordinal μ (of μ^-), $\kappa(\mu)$ is a strong limit cardinal of cofinality $> \kappa(\mu^-)$. For limit ordinals, $\kappa(\mu)$ is the union of all previous $\kappa(\mu')$. Let ω_1 be the first uncountable ordinal. We claim that $\kappa(\omega_1)$ has the desired properties.

Indeed, any $\lambda < \kappa(\omega_1)$ is also less than $\kappa(\mu)$ for some $\mu < \omega_1$, which we may assume to be a successor ordinal, and so $2^\lambda < \kappa(\mu) < \kappa(\omega_1)$; thus, $\kappa(\omega_1)$ is a strong limit cardinal. Also, ω_1 has uncountable cofinality, so $\kappa(\omega_1)$ has uncountable cofinality. Finally, for all $\lambda < \kappa(\omega_1)$, one has $\lambda < \kappa(\mu)$ for some $\mu < \omega_1$, and then for all successor ordinals $\mu' > \mu$ less than ω_1, the cardinal $\kappa(\mu') < \kappa(\omega_1)$ is by construction a strong limit cardinal with cofinality greater than λ. □

Convention 4.2. From now on, we fix a cardinal κ as in Lemma 4.1, and all of our perfectoid spaces will be assumed to be κ-small in the following sense.

Definition 4.3. *A perfectoid space X is κ-small if the cardinality of $|X|$ is less than κ, and for all open affinoid subspaces $U = \mathrm{Spa}(A, A^+) \subset X$, the cardinality of A is less than κ.*

p. 38 [Hub96]

For the results of this section, it is actually enough to assume that κ is an uncountable strong limit cardinal.

Remark 4.4. If A has a dense subalgebra $A_0 \subset A$ of cardinality less than κ, then A has cardinality less than κ. Indeed, identifying elements of A with convergent Cauchy sequences in A_0, one sees that if λ is the cardinality of A_0, then the cardinality of A is at most

$$\lambda^\omega \leq (2^\lambda)^\omega = 2^{\lambda \times \omega} = 2^\lambda < \kappa .$$

Proposition 4.5. *A perfectoid space X is κ-small if and only if X admits a cover by less than κ many open affinoid subspaces $U = \mathrm{Spa}(A, A^+)$ for which the cardinality of A is less than κ.*

[Hub94] p. 518

In particular, the category of κ-small perfectoid spaces is stable under finite products.

Proof. Assume that X is κ-small. In that case, the set of all subsets of X is of cardinality less than κ, and in particular X is covered by less than κ many open subspaces of the form $U = \mathrm{Spa}(A, A^+)$, where the cardinality of A is less than κ.

open affinoid subspace p. 46 [Hub96]

Conversely, for any open affinoid subspace $U = \mathrm{Spa}(A, A^+) \subset X$, the space $|U|$ is a subset of the space of binary relations on A (where $x \in U$ corresponds to the relation $|a(x)| \leq |b(x)|$ for $a, b \in A$). Thus, U embeds into the power set of $A \times A$, which is of cardinality $2^{|A|^2} = 2^{|A|} < \kappa$. If X is covered by less than κ many such U, then also the cardinality of $|X|$ is less than κ. Also, for any open affinoid subspace $U = \mathrm{Spa}(A, A^+)$ of X, one can (by quasicompacity) embed U into a union of finitely many rational subsets of $V = \mathrm{Spa}(B, B^+)$ for which B has cardinality less than κ.

p. 37-40 [Hub96]

Def. 1.1.4 [Hub 98]

are quasi-compact p. 39 [Hub96]

p. 38 [Hub96]

Each of the rational subsets is of the form $\mathrm{Spa}(B', B'^+)$, where B' has a dense subalgebra which is finitely generated over B; in particular, the cardinality of B' is also less than κ. As then A embeds into the finite product of the B''s, one also sees that the cardinality of A is less than κ. □

5. Morphisms of perfectoid spaces

In this section, we define some basic classes of maps between perfectoid spaces.

Definition 5.1. *A map $f : Y \to X$ of perfectoid spaces is an injection if for all perfectoid spaces Z, the map $f_* : \mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, X)$ is injective.*

We remind the reader that Convention 4.2 is in place; this means that implicitly, we assume that X and Y are κ-small, and the condition ranges over all κ-small Z. We note that this condition could therefore a priori depend on the choice of κ; however, it follows from Proposition 5.3 (specifically, condition (iii)) that it does not. Similar remarks apply to all other definitions below.

It is enough to check the condition for affinoid perfectoid Z.

Example 5.2. Let X be a perfectoid space, and $x \in X$ a point, giving rise to the map

$$i_x : \mathrm{Spa}(K(x), K(x)^+) \to X ,$$

where $K(x)$ is the completed residue field at x, and $K(x)^+ \subset K(x)$ the corresponding open and bounded valuation subring. Then the map i_x is an injection. To check this, we may assume that X is affinoid by replacing X by an affinoid neighborhood of x. Now

$$|\mathrm{Spa}(K(x), K(x)^+)| = \bigcap_{U \ni x} U \subset X ,$$

where the intersection runs over all rational neighborhoods U of x in X. In fact, $\mathrm{Spa}(K(x), K(x)^+)$ is the inverse limit of U (considered as a perfectoid space) in the category of perfectoid spaces, as

$$K(x)^+/\varpi = \varinjlim_{U \ni x} \mathcal{O}_X^+(U)/\varpi , \tag{1}$$

and so $K(x)^+$ is the ϖ-adic completion of $\varinjlim_{U \ni x} \mathcal{O}_X^+(U)$. As each $U \to X$ is an injection, it follows that i_x is an injection.

In particular, note that if X is qcqs and has a unique closed point $x \in X$, then $X = \mathrm{Spa}(K(x), K(x)^+)$, as in this case X is the only quasicompact open subset of X containing x.

We have the following characterizations of injections.

Proposition 5.3. *Let $f : Y \to X$ be a map of perfectoid spaces. The following conditions are equivalent.*

(i) *The map f is an injection.*

(ii) *For all perfectoid fields K with an open and bounded valuation subring $K^+ \subset K$, the map $f_* : Y(K, K^+) \to X(K, K^+)$ is injective.*

(iii) *The map $|f| : |Y| \to |X|$ is injective, and for all rank-1-points $y \in Y$ with image $x = f(y) \in X$, the map of completed residue fields $K(x) \to K(y)$ is an isomorphism.*

(iv) *The map $|f| : |Y| \to |X|$ is injective, and the map $f : Y \to X$ is final in the category of maps $g : Z \to X$ for which $|g| : |Z| \to |X|$ factors over a continuous map $|Z| \to |Y|$.*

In particular, by (iv), given X, injections $Y \hookrightarrow X$ of perfectoid spaces are determined by the injective map of topological spaces $|Y| \to |X|$. We warn the reader that in general $|Y|$ may not have the subspace topology from $|X|$.

Proof. Clearly, (i) implies (ii). Next, we check that (ii) implies (iii). To see that $|f|$ is injective, assume that two points $y_1, y_2 \in Y$ map to $x \in X$. We may replace X by $\mathrm{Spa}(K, K^+)$ via base change along i_x as in Example 5.2. There exists an extension $(K, K^+) \subset (L, L^+)$ of affinoid perfectoid fields such that there are (L, L^+)-valued points $\widetilde{y}_1, \widetilde{y}_2 \in Y(L, L^+)$ which send the closed point of $\mathrm{Spa}(L, L^+)$ to y_1 resp. y_2, and with same image in $X(L, L^+)$ given by the map $(K, K^+) \to (L, L^+)$. As f satisfies (ii), this implies $\widetilde{y}_1 = \widetilde{y}_2$, and thus $y_1 = y_2$.

Now given a rank-1-point $y \in Y$ with image $x \in X$, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Spa}(K(y), \mathcal{O}_{K(y)}) & \xrightarrow{i_y} & Y \\ \downarrow & & \downarrow f \\ \mathrm{Spa}(K(x), \mathcal{O}_{K(x)}) & \xrightarrow{i_x} & X. \end{array}$$

It follows that $g : \mathrm{Spa}(K(y), \mathcal{O}_{K(y)}) \to \mathrm{Spa}(K(x), \mathcal{O}_{K(x)})$ is functorially injective on (K, K^+)-valued points for all (K, K^+). We claim that the two maps

$$K(y) \to K(y) \widehat{\otimes}_{K(x)} K(y)$$

agree. Indeed, they agree after composition with $K(y)\widehat{\otimes}_{K(x)}K(y) \to L$ for any perfectoid field L by assumption; but as $K(y)\widehat{\otimes}_{K(x)}K(y)$ is perfectoid (in particular, uniform), no element lies in the kernel of all maps to perfectoid fields. This implies that the surjection $K(y)\widehat{\otimes}_{K(x)}K(y) \to K(y)$ is an isomorphism. It follows that also the base change $K(y) \to K(y)\widehat{\otimes}_{K(x)}K(y)$ of $K(x) \to K(y)$ is an isomorphism. Let $V = K(y)/K(x)$, which is a $K(x)$-Banach space; then (by exactness of $\widehat{\otimes}_{K(x)}K(y)$), we have $V\widehat{\otimes}_{K(x)}K(y) = 0$, which implies that $V = 0$. Thus, $K(y) = K(x)$, as desired, finishing the proof that (ii) implies (iii).

To check that (iii) implies (iv), we first check that for all perfectoid fields K with an open and bounded valuation subring $K^+ \subset K$, the map $Y(K, K^+) \to X(K, K^+)$ is injective, with image those maps $\mathrm{Spa}(K, K^+) \to X$ that factor over $|Y|$. Thus, assume given a map $\mathrm{Spa}(K, K^+) \to X$. The image of the closed point is a point $x \in X$, and $\mathrm{Spa}(K, K^+)$ factors uniquely over the injection $i_x : \mathrm{Spa}(K(x), K(x)^+) \to X$. Replacing X by $\mathrm{Spa}(K(x), K(x)^+)$ and Y by $Y \times_X \mathrm{Spa}(K(x), K(x)^+)$, we may assume that $X = \mathrm{Spa}(K(x), K(x)^+)$. Now $|Y| \subset |X|$ is a subset stable under generalizations; in particular, any quasicompact open subset of Y has a unique closed point, so Y is an increasing union of open subfunctors of the form $\mathrm{Spa}(K(x), (K(x)^+)') \subset \mathrm{Spa}(K(x), K(x)^+)$, for varying $(K(x)^+)'$. But it is easy to see that for any such $(K(x)^+)'$,

$$\mathrm{Spa}(K(x), (K(x)^+)')(K, K^+) \to \mathrm{Spa}(K(x), K(x)^+)(K, K^+)$$

is injective, with image those maps that factor through $|\mathrm{Spa}(K(x), (K(x)^+)')| \subset |\mathrm{Spa}(K(x), K(x)^+)|$; thus, the same holds for Y.

Now take any map $g : Z \to X$ such that $|g|$ factors continuously over $|Y| \subset |X|$. We need to check that g factors uniquely over a map $h : Z \to Y$. This can be done locally on Z. Given a point $z \in Z$, we can replace Z by a small open affinoid neighborhood of z so that $|Z| \to |X|$ factors over an affinoid open subset of X, and also the map $|Z| \to |Y|$ factors over an affinoid open subset of Y.

Thus, we can assume that X, Y and Z are all affinoid. Now consider the qcqs map $Y \times_X Z \to Z$. By Lemma 5.4 below, it is enough to check that $(Y \times_X Z)(K, K^+) \to Z(K, K^+)$ is bijective for all perfectoid fields K with an open and bounded valuation subring $K^+ \subset K$. But this follows from the description of the map $Y(K, K^+) \to X(K, K^+)$ above.

Finally, (iv) implies (i): Given any map $Z \to X$, we need to see that there is at most one way to factor it over Y. If $|Z| \to |X|$ does not factor continuously over $|Y|$, there is no such factorization. However, if $|Z| \to |X|$ factors continuously over $|Y|$, then there is a unique factorization over Y, by the universal property of $Y \to X$ in (iv). □

Lemma 5.4. *Let $f : Y \to X$ be a qcqs map of perfectoid spaces. The following conditions are equivalent.*

(i) *The map f is an isomorphism.*

(ii) *The map $|f| : |Y| \to |X|$ is bijective, and for all rank-1-points $y \in Y$ with image $x \in X$, the induced map of residue fields $K(x) \to K(y)$ is an isomorphism.*

(iii) *For all perfectoid fields K with open and bounded valuation subring $K^+ \subset K$, the map $Y(K, K^+) \to X(K, K^+)$ is a bijection.*

(iv) *For all algebraically closed perfectoid fields C with an open and bounded valuation subring $C^+ \subset C$, the map $Y(C, C^+) \to X(C, C^+)$ is a bijection.*

[Huber 94] Prop. 4.3, p. 541

Proof. Clearly (i) implies (ii) and (iii). First, we check that (ii) and (iii) are equivalent. To see that (ii) implies (iii), take any map $\mathrm{Spa}(K, K^+) \to X$. Let $x \in X$ be the image of the closed point; then the map factors over the immersion $i_x : \mathrm{Spa}(K(x), K(x)^+) \to X$. We can replace X by $\mathrm{Spa}(K(x), K(x)^+)$. Now Y has a unique closed point $y \in Y$, so that $Y = \mathrm{Spa}(K(y), K(y)^+)$. By the assumption in (ii), we have $K(x) = K(y)$, and then bijectivity forces $K(y)^+ = K(x)^+$. Now it is clear that (iii) holds.

For the converse (iii) implies (ii), assume that (iii) holds. To show that this implies (ii), it suffices to show that for any $x \in X$, the map $Y \times_X \mathrm{Spa}(K(x), K(x)^+) \to \mathrm{Spa}(K(x), K(x)^+)$ is an isomorphism. We may assume that $X = \mathrm{Spa}(K(x), K(x)^+)$. The condition in (iii) gives us a unique section $X \to Y$. Let $y \in Y$ be the image of the closed point $x \in X$. Note that the immersion $i_y : Y' = \mathrm{Spa}(K(y), K(y)^+) \to Y$ has the property that $Y' \to X$ still satisfies the conditions of (iii): Indeed, $Y'(K, K^+) \subset Y(K, K^+) = X(K, K^+)$ is certainly injective, but there is a section $X \to Y'$, so that it is also surjective. In particular, any point of Y lies in Y', so that in fact $Y = Y'$. As $Y \to X$ has a section, we get maps $K(x) \to K(y) \to K(x)$ of fields whose composite is the identity; as maps of fields are injective, this implies $K(y) = K(x)$, and then also $K(y)^+ = K(x)^+$ as $K(x)^+ \subset K(y)^+ \subset K(x)^+$. Thus, $Y \cong X$, as desired.

Now we show that (ii) and (iii) together imply (i). We can assume that X is affinoid; then, by assumption on f, Y is qcqs. Then $|f| : |Y| \to |X|$ is a bijective, generalizing and spectral map of spectral spaces. By Lemma 2.5, it is a quotient map, and thus a homeomorphism. It remains to see that $\mathcal{O}_X^+ \to \mathcal{O}_Y^+$ is an isomorphism of sheaves, for which it is enough to show that $\mathcal{O}_X^+/\varpi \to \mathcal{O}_Y^+/\varpi$ is an isomorphism of sheaves, where ϖ is a pseudouniformizer on X. We can check this on stalks. But the stalk of $\mathcal{O}_X^+/\varpi$ at $x \in X$ is given by $K(x)^+/\varpi$, and similarly for $\mathcal{O}_Y^+/\varpi$. As $K(x)^+/\varpi = K(y)^+/\varpi$ (by the proof that (iii) implies (ii)), the result follows.

Clearly, (iii) implies (iv), so it remains to prove that (iv) implies (iii). Note that injectivity of $Y(K, K^+) \to X(K, K^+)$ is automatic by embedding (K, K^+) into an algebraic closure (C, C^+). But in fact, given any map $\mathrm{Spa}(K, K^+) \to X$, we can lift to a map $\mathrm{Spa}(C, C^+) \to Y$, which factors over

some affinoid $\mathrm{Spa}(S, S^+) \subset Y$, and is given by a map $(S, S^+) \to (C, C^+)$. But this map has to be Galois-equivariant, and thus factors over (K, K^+). This gives the desired lift $\mathrm{Spa}(K, K^+) \to Y$. □

Moreover, injections have a simple behaviour on pullbacks.

Corollary 5.5. (i) *Let $f : Y \to X$ be an injection of perfectoid spaces, and $X' \to X$ any map of perfectoid spaces. Then the pullback $f' : Y' = X' \times_X Y \to X'$ is an injection, and the map*

$$|Y'| \to |X'| \times_{|X|} |Y|$$

is a homeomorphism.

(ii) *Let $f : Y \to X$ be a map of perfectoid spaces. Then f is an injection if and only if f is universally injective, i.e. for all maps $X' \to X$ with pullback $f' : Y' = X' \times_X Y \to X'$, the map $|f'| : |Y'| \to |X'|$ is injective.*

Proof. In part (i), it is clear from the definition that f' is an injection. Using that $|f|$ and $|f'|$ are injective by Proposition 5.3, it follows that $|Y'| \to |X'| \times_{|X|} |Y|$ is injective. Moreover, all maps of perfectoid spaces are generalizing, so both $|Y'|$ and $|X'| \times_{|X|} |Y|$ are generalizing subsets of $|Y|$, and thus the map $|Y'| \to |X'| \times_{|X|} |Y|$ is generalizing. But this map is always surjective and spectral. By Lemma 2.5, it is a homeomorphism.

For part (ii), we have seen that if f is an injection, then it is universally injective. For the converse, assume that f is universally injective. Given any map $g : Z \to X$, we need to see that there is at most one way to factor it over $f : Y \to X$. Pulling back by g, we may assume that $Z = X$, and we have to show that there is at most one section of f. Assume that there is a section $h : X \to Y$. Being a section, h is an injection; but then $|X| \to |Y| \to |X|$ is a factorization of the identity as a composition of injective maps. Thus, $|Y| = |X|$, and the map is a homeomorphism; then, by Proposition 5.3, it follows that $Y = X$. In particular, h is the only section, as desired. □

As injections are determined by their behaviour on topological spaces, the following definition is reasonable.

Definition 5.6. *A map $f : Y \to X$ of perfectoid spaces is an immersion if f is an injection and $|f| : |Y| \to |X|$ is a locally closed immersion. If $|f|$ in addition is open (resp. closed), then f is an open (resp. closed) immersion.*

One cannot define closed immersions of schemes or adic spaces this way because of possible non-reduced structures. This is not a concern for perfectoid spaces, which makes this general definition possible.

Zariski closed embeddings as defined in [Sch15, Section II.2] are a special class of closed embeddings.

Definition 5.7 ([Sch15, Definition II.2.1, Definition II.2.6])**.** *Let $f : Z \to X$ be a map of perfectoid spaces, where $X = \mathrm{Spa}(R, R^+)$ is affinoid.*

(i) *The map f is Zariski closed if f is a closed immersion and there is an ideal $I \subset R$ such that $|Z| \subset |X|$ is the locus where $|f| = 0$ for all $f \in I$.*

(ii) *The map f is strongly Zariski closed if $Z = \mathrm{Spa}(S, S^+)$ is affinoid, $R \to S$ is surjective, and S^+ is the integral closure of R^+ in S.*

By [KL15, Proposition 3.6.9 (c)], if f is strongly Zariski closed, then $R^+ \to S^+$ is almost surjective, so that the definition agrees with [Sch15, Definition II.2.6].

$$\varinjlim_{U} H^n(U, F) \xrightarrow{\sim} H^n(\overline{V}, F|\overline{V})$$

where U runs through all open neighbourhoods of $\overline{V}$ in X.

Proof. Let $(U_i | i \in I)$ be the family of all quasi-compact open neighbourhoods of $\overline{V}$ in X. $\overline{V}$ is the set of the specializations of the points of V in X. Hence by (1.1.10.i) $\overline{V}$ is closed under generalizations in X. Therefore $\overline{V} = \bigcap_{i \in I} U_i$, and thus $(X, \overline{V}) \sim \varprojlim_i (X, U_i)$. □

2.5 Strict localizations of adic spaces

Definition 2.5.1. i) A *geometric point* (in our category of prepseudo-adic spaces) is a pseudo-adic space S such that $\underline{S}$ is the spectrum of a separably algebraically closed affinoid field ((1.1.5)) and $|S| = \{s\}$ where s is the closed point of $\underline{S}$ ((1.1.6)).
ii) Let X be a prepseudo-adic space. A *geometric point of* X is a morphism of prepseudo-adic spaces $u: \xi \to X$ where ξ is a geometric point. The point $u(|\xi|) \in |X|$ is called the *support* of ξ and is denoted by $\mathrm{supp}(\xi)$.

(2.5.2) Let X be a prepseudo-adic space. For every $x \in |X|$, we define a geometric point $\overline{x}$ of X as follows. Let $\overline{k}(x)$ be a separable algebraic closure of $k(x)$, and let $\overline{k}(x)^+$ be a valuation ring of $\overline{k}(x)$ which extends $k(x)^+$ (all valuation rings of $\overline{k}(x)$ which extend $k(x)^+$ are conjugated [B, VI.8.6 Cor.1]). $\overline{k}(x)^+$ is microbial if $k(x)^+$ is microbial. We equip $\overline{k}(x)$ with a topology, namely if $k(x)$ is a non-archimedian field then we equip $\overline{k}(x)$ with the valuation topology of $\overline{k}(x)^+$ and if the topology of $k(x)$ is discrete then we equip $\overline{k}(x)$ with the discrete topology. Then

$$\overline{\kappa}(x) := (\overline{k}(x), \overline{k}(x)^+)$$

is a separably algebraically closed affinoid field. We put

$$\overline{x} := (\mathrm{Spa}\overline{\kappa}(x), \{s\})$$

where s is the closed point of $\mathrm{Spa}\overline{\kappa}(x)$. By (1.1.8) the inclusion $k(x) \to \overline{k}(x)$ induces a morphism of prepseudo-adic spaces

$$\overline{x} \to X$$

with support x.

For every geometric point ξ, the global section functor

$$\Gamma(\xi, -): \xi_{\mathrm{et}}^{\sim} \longrightarrow (\mathrm{sets})$$

(A.5.d): Let be given an object U of $X_{\acute{e}t,g}$, an object $i \in I$, an object V of $X_{i,\acute{e}t\,fp}$ and a morphism $\mu: U \to V \times_X X$ in $X_{\acute{e}t,g}$. Choosing a covering of U by quasi-compact pseudo-adic spaces, we may assume that U is quasi-compact. Then $\mathrm{im}(|\mu|)$ is a quasi-compact open subset of $|V \times_X X|$. By (2.4.3.ii) we may assume that there exists a quasi-compact open pseudo-adic subspace V' of V such that μ factors through a morphism $\mu': U \to V' \times_X X$ with $\mathrm{im}(|\mu'|) = |V' \times_X X|$. μ' is a covering and an epimorphism in $X_{\acute{e}t,g}$ ((2.3.5)). This shows that λ satisfies (A.5.d). □

Corollary 2.4.6. *Let $p: I \to \mathcal{A}$, $c: I \to \mathcal{A}$ and $\varphi = (\varphi_i)_{i \in I}: c \to p$ be as in (2.4.1). Assume that I has a final object 0, and let F_0 be a sheaf of sets (resp. groups, resp. abelian groups) on $X_{0,\acute{e}t}$. Let $F \in X^{\sim}_{\acute{e}t}$ be the preimage of F_0 on $X_{\acute{e}t}$ and, for every $i \in I$, let $F_i \in X^{\sim}_{i,\acute{e}t}$ be the preimage of F_0 on $X_{i,\acute{e}t}$.*
If $\varphi: X \sim \varprojlim_i X_i$ then, for $n = 0$ (resp. $n \in \{0,1\}$, resp. $n \in \mathbb{N}_0$), the natural mapping

$$\varinjlim_i H^n(X_i, F_i) \longrightarrow H^n(X, F)$$

is bijective.

Proof. For abelian sheaves F_0 the assertion follows from [SGA 4, VI.8.7.7], (2.4.4) and (2.3.13). For $n \in \{0,1\}$ one can prove (2.4.6) by a direct computation. □

Example 2.4.7. Let X be an adic space, Y a closed adic subspace of X and F a sheaf of sets (resp. groups, resp. abelian groups) on $X_{\acute{e}t}$. Assume that X is analytic and quasi-separated and there is a locally finite covering of X by quasi-compact open subsets of X. Then, for $n = 0$ (resp. $n \in \{0,1\}$, resp. $n \in \mathbb{N}_0$),

$$\varinjlim_U H^n(U, F) \xrightarrow{\sim} H^n(Y, F|Y)$$

where U runs through all open neighbourhoods of Y in X.

Proof. We prove (2.4.7) in the special case that X is quasi compact. Then the general case follows by standard arguments. Let $(U_i | i \in I)$ be the family of all quasi-compact open neighbourhoods of Y in X. Since Y is closed under generalizations in X, we have $Y = \bigcap_{i \in I} U_i$. Hence $Y \sim \varprojlim_i U_i$. □

Example 2.4.8. Let X be a quasi-compact quasi-separated analytic adic space, V a quasi-compact open subset of X, and F a sheaf of sets (resp. groups, resp. abelian groups) on $X_{\acute{e}t}$. Let $\bar{V}$ be the closure of V in X and let $F|\bar{V}$ denote the restriction of F to the étale site of the pseudo-adic space $\bar{V} = (X, \bar{V})$. Then, for $n = 0$ (resp. $n \in \{0,1\}$, resp. $n \in \mathbb{N}_0$),

is an equivalence of categories ((1.7.3) or (2.3.10)).

Definition 2.5.3. Let $u:\xi \to X$ be a geometric point of a prepseudo-adic space X. For every sheaf F on $X_{\text{ét}}$, we put

$$F_\xi := \Gamma(\xi, u^* F)$$

and call F_ξ the *stalk* of F at ξ.

If ξ_1, ξ_2 are geometric points of a prepseudo-adic space X with $\mathrm{supp}(\xi_1) = \mathrm{supp}(\xi_2)$ then the stalk functors $X_{\text{ét}}^{\sim} \to (\text{sets})$, $F \mapsto F_{\xi_i} (i = 1, 2)$ are isomorphic.

Lemma 2.5.4. *For a geometric point $u:\xi \to X$ of a prepseudo-adic space X, let C_ξ be the category of ξ-pointed prepseudo-adic spaces étale over X, i.e. C_ξ is the category of pairs (V, v) where $V \in \mathrm{ob}\acute{\mathrm{E}}\mathrm{t}/X$ and $v:\xi \to V$ is a X-morphism. The category C_ξ is cofiltered.*
For every presheaf P on $X_{\text{ét}}$, there is a functorial bijection

$$(aP)_\xi \simeq \varinjlim_{(V,v)\in C_\xi} P(V),$$

where aP denotes the sheaf associated with P.

Proof. C_ξ is cofiltered, since fibre products exist in C_ξ. Let Q be the presheaf preimage of P on $\xi_{\text{ét}}$ and let aQ be the associated sheaf. By (1.7.3) the covering $\mathrm{id}:\xi \to \xi$ of ξ refines every covering of ξ in $\xi_{\text{ét}}$. Hence $\Gamma(\xi, Q) \xrightarrow{\sim} \Gamma(\xi, aQ)$. By [SGA 4, III.1.3] $u^*(aP) = aQ$. Therefore $(aP)_\xi = \Gamma(\xi, u^*(aP)) = \Gamma(\xi, Q) = \varinjlim_{(V,v)\in C_\xi} P(V)$. □

Proposition 2.5.5. *Let X be a prepseudo-adic space. Then the family of functors $(X_{\text{ét}}^{\sim} \to (\text{sets}),\ F \mapsto F_{\overline{x}})_{x\in|X|}$ is conservative.*

Proof. The assertion follows from (2.5.4), since by (1.7.3), for every $(h:Y \to X) \in \mathrm{ob}\acute{\mathrm{E}}\mathrm{t}/X$ and $y \in |Y|$, $x := h(y) \in |X|$, there exists a X morphism $v:\overline{x} \to Y$ with $\{y\} = \mathrm{im}(|v|)$. □

Next we want to define strictly local prepseudo-adic spaces.

Lemma 2.5.6. *For a prepseudo-adic space X, the following conditions are equivalent*
i) *There exists a $x \in |X|$ such that, for every étale morphism of prepseudo-adic spaces $f:Y \to X$ and every $y \in |Y|$ with $x = f(y)$, there is an open prepseudo-adic subspace U of Y such that $y \in |U|$ and the restriction $f:U \to X$ is an isomorphism.*
ii) *X has a unique closed point x (i.e., every point of X specializes to x), $x \in |X|$ and $\mathcal{O}_{X,x}$ is a strictly local ring.*
iii) *$\underline{X}$ is the spectrum of an affinoid ring (A, A^+) with A a strictly local ring, $\underline{X}$ has a unique closed point x, the maximal ideal of A is the support of x, and $x \in |X|$.*

Proof. Assume (i). Then every point of $\underline{X}$ specializes to x (since otherwise there would exist an open prepseudo-adic subspace Y of X with $x \in |Y|$ and $\underline{Y} \neq \underline{X}$). Hence $\underline{X}$ is affinoid. Put $A := \mathcal{O}_{\underline{X}}(\underline{X}) = \mathcal{O}_{\underline{X},x}$. Let $g: Z \to \operatorname{Spec} A$ be an étale morphism of schemes, and let z be a point of Z lying over the closed point of $\operatorname{Spec} A$. Consider

$$\begin{array}{ccc} \underline{Y} := \underline{X} \times_{\operatorname{Spec} A} Z & \xrightarrow{p} & Z \\ \downarrow q & & \downarrow g \\ \underline{X} & \xrightarrow[r]{} & \operatorname{Spec} A \end{array}$$

where r is induced by the identity $A \to A$. Choose a point $y \in \underline{Y}$ with $p(y) = z$ and $q(y) = x$. By i) the ring homomorphism $q_y^*: A \to \mathcal{O}_{\underline{Y},y}$ is bijective. Let $\varphi: \mathcal{O}_{Z,z} \to A$ be the composition of $p_y^*: \mathcal{O}_{Z,z} \to \mathcal{O}_{\underline{Y},y}$ and $(q_y^*)^{-1}: \mathcal{O}_{\underline{Y},y} \to A$. Then φ defines a section $h: \operatorname{Spec} A \to Z$ of g. Hence by [EGA, IV.18.8.1] A is strictly local. Therefore (ii) is satisfied.

Assume (ii). Then $\underline{X}$ is affinoid and $\mathcal{O}_{\underline{X}}(\underline{X}) = \mathcal{O}_{\underline{X},x}$. Hence (iii) is satisfied.

(iii) implies (i) by (1.7.3.ii). □

Remark 2.5.7. Let X be the spectrum of an affinoid ring (A, A^+) with A a local ring. Let x be a point of X whose support is the maximal ideal of A. Then x is the unique closed point of X if and only if $A^+ = \{a \in A \mid |a(x)| \leq 1\}$.

Proof. Assume that x is the unique closed point of X, i.e. every point of X specializes to x. Put $B := \{a \in A \mid |a(x)| \leq 1\}$. Then $B = \{a \in A \mid |a(z)| \leq 1 \text{ for every } z \in X\} = A \cap \mathcal{O}_X^+(X) = A^+$.

Now let us assume that $A^+ = \{a \in A \mid |a(x)| \leq 1\}$. Let z be a point of X. We show that z is a generalization of x. For that let $U := \{y \in X \mid |a(y)| \leq |b(y)| \neq 0\}$ $(a, b \in A)$ be an open subset of X which contains x. Then b is a unit of A (since $b(x) \neq 0$) and $|\frac{a}{b}(x)| \leq 1$, i.e. $\frac{a}{b} \in A^+$. Therefore $|\frac{a}{b}(z)| \leq 1$, i.e. $z \in U$. □

Definition 2.5.8. A prepseudo-adic space X is called *strictly local* if X satisfies the equivalent conditions of (2.5.6).

(2.5.9) Let X be a prepseudo-adic space. For every geometric point ξ of X, we define a strictly local prepseudo-adic $X(\xi)$ as follows. Let D_ξ denote the full subcategory of C_ξ ((2.5.4)) consisting of those objects $(V, v) \in C_\xi$ such that $\underline{V}$ is affinoid. Then D_ξ is cofinal in C_ξ. Put

$$\mathcal{O}_{X,\xi} := \varinjlim_{(V,v) \in C_\xi} \mathcal{O}_{\underline{V}}(\underline{V}) = \varinjlim_{(V,v) \in D_\xi} \mathcal{O}_{\underline{V}}(\underline{V})$$

$$\mathcal{O}_{X,\xi}^+ := \varinjlim_{(V,v) \in C_\xi} \mathcal{O}_{\underline{V}}^+(\underline{V}) = \varinjlim_{(V,v) \in D_\xi} \mathcal{O}_{\underline{V}}^+(\underline{V}).$$

If (V, v) is an object of D_ξ and I an ideal of definition of a ring of definition of $\mathcal{O}_{\underline{V}}(\underline{V})$ then the group topology of $\mathcal{O}_{X,\xi}$ such that $\{I^n \cdot \mathcal{O}^+_{X,\xi} \mid n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 is independent of the choice of (V, v) and I. We equip $\mathcal{O}_{X,\xi}$ with this topology. Then $(\mathcal{O}_{X,\xi}, \mathcal{O}^+_{X,\xi})$ is an affinoid ring (we will show in (2.5.13) that $\mathcal{O}_{X,\xi}$ satisfies one of the conditions of (1.1.1)). Put

$$X(\xi)_- := \mathrm{Spa}(\mathcal{O}_{X,\xi}, \mathcal{O}^+_{X,\xi}).$$

For every $(V, v) \in \mathrm{ob} D_\xi$, let

$$\underline{\varphi}_{(V,v)} : X(\xi)_- \longrightarrow \underline{V}$$

be the morphism of adic spaces given by the natural ring homomorphism $(\mathcal{O}_{\underline{V}}(\underline{V}), \mathcal{O}^+_{\underline{V}}(\underline{V})) \to (\mathcal{O}_{X,\xi}, \mathcal{O}^+_{X,\xi})$. Put

$$|X(\xi)| := \bigcap_{(V,v) \in D_\xi} \underline{\varphi}^{-1}_{(V,v)}(|V|).$$

Then

$$X(\xi) := (X(\xi)_-, |X(\xi)|)$$

is a prepseudo-adic space, and, for every $(V, v) \in \mathrm{ob} D_\xi$, $\underline{\varphi}_{(V,v)}$ gives a morphism of prepseudo-adic spaces

$$\varphi_{(V,v)} : X(\xi) \longrightarrow V.$$

There is a natural morphism

$$X(\xi) \longrightarrow X$$

and a natural X-morphism

$$\xi \longrightarrow X(\xi).$$

If ξ_1, ξ_2 are two geometric points of X with $\mathrm{supp}(\xi_1) = \mathrm{supp}(\xi_2)$ then $X(\xi_1)$ and $X(\xi_2)$ are X-isomorphic.

Lemma 2.5.10. *We remain in the situation of* (2.5.9). *Let $t \in |X(\xi)|$ be the image point of $|\xi|$ under the natural morphism $\xi \to X(\xi)$.*
Then $\mathcal{O}_{X,\xi}$ is a strictly local ring, t is the unique closed point of $X(\xi)_-$ and the maximal ideal of $\mathcal{O}_{X,\xi}$ is the support of t. Hence $X(\xi)$ is strictly local.

Proof. Let s be the point of $|\xi|$. Then $\mathcal{O}_{X,\xi} = \varinjlim_{(W,w)\in D_\xi} \mathcal{O}_{\underline{W}}(\underline{W}) = \varinjlim_{(W,w)\in D_\xi} \mathcal{O}_{\underline{W},w(s)}$. Hence $\mathcal{O}_{X,\xi}$ is a local ring. t is the valuation of $\mathcal{O}_{X,\xi}$ whose restriction to each $\mathcal{O}_{\underline{W},w(s)}$ is $v_{w(s)}$. The support of $v_{w(s)}$ is the maximal ideal of $\mathcal{O}_{\underline{W},w(s)}$, and therefore the support of t is the maximal ideal of $\mathcal{O}_{X,\xi}$. Since $\mathcal{O}^+_{X,\xi} = \varinjlim_{(W,w)\in D_\xi} \mathcal{O}^+_{\underline{W}}(\underline{W}) = \varinjlim_{(W,w)\in D_\xi} \mathcal{O}^+_{\underline{W},w(s)}$ and $\mathcal{O}^+_{\underline{W},w(s)} = \{f \in \mathcal{O}_{\underline{W},w(s)} \mid v_{w(s)}(f) \leq 1\}$ ([Hu, 1.6.ii]), we obtain $\mathcal{O}^+_{X,\xi} = \{f \in \mathcal{O}_{X,\xi} \mid t(f) \leq 1\}$. Hence by (2.5.7) t is the unique closed point of $X(\xi)_-$. (1.7.3.i) and the characterization of strictly local rings in [EGA, IV.18.8.1.c] imply that $\mathcal{O}_{X,\xi}$ is strictly local. □

Definition 2.5.11. Let X be a prepseudo-adic space and ξ a geometric point of X. The strictly local prepseudo-adic space $X(\xi)$ is called the *strict localization of X at ξ*.

Obviously we have (with the notation of (2.4.2))

Lemma 2.5.12. *Let X be a pseudo-adic space and ξ a geometric point of X. Let E_ξ be the full subcategory of D_ξ ((2.5.9)) consisting of those objects $(V, v) \in D_\xi$ such that $|V|$ is quasi-compact. Then E_ξ is cofinal in D_ξ, and the morphisms*

$$\varphi_{(V,v)}: X(\xi) \longrightarrow V$$

give

$$\varphi: X(\xi) \sim \varprojlim_{(V,v)\in E_\xi} V.$$

In the following proposition we give an explicit description of the strict localizations of prepseudo-adic spaces.

Proposition 2.5.13. *Let X be a prepseudo-adic space, $\xi \to X$ a geometric point of X and $x \in |X|$ the support of ξ.*

i) Assume that x is an analytic point of $\underline{X}$. Let $p: \mathrm{Spa}\overline{\kappa}(x) \to \underline{X}$ be the natural morphism from $\mathrm{Spa}\overline{\kappa}(x)$ to $\underline{X}$ ((2.5.2)). Then the completion of $(\mathcal{O}_{X,\xi}, \mathcal{O}^+_{X,\xi})$ is isomorphic to the completion of $\overline{\kappa}(x)$, and $X(\xi)$ is X-isomorphic to the prepseudo-adic space $(\mathrm{Spa}\overline{\kappa}(x), p^{-1}(|X|))$.

ii) Assume that x is a non-analytic point of $\underline{X}$. Let $U = \mathrm{Spa}(A, A^+)$ be an affinoid open neighbourhood of x in $\underline{X}$ such that A has a noetherian ring of definition, let B be the strict henselization of A at $\mathrm{supp}(x) \in \mathrm{Spec}A$, let y be a valuation of B which extends the valuation x of A, put $B^+ := \{b \in B \mid y(b) \leq 1\}$, and equip B with the $I \cdot B$-adic topology where I is an ideal of definition of a ring of definition of A. Then (B, B^+) is an affinoid ring. Let $p: \mathrm{Spa}(B, B^+) \to \underline{X}$ be the morphism of adic spaces which is induced by the ring homomorphism $A \to B$.

Then the completion of $(\mathcal{O}_{X,\xi}, \mathcal{O}^+_{X,\xi})$ is isomorphic to the completion of (B, B^+), and $X(\xi)$ is X-isomorphic to the prepseudo-adic space $(\mathrm{Spa}(B, B^+), p^{-1}(|X|))$.

Proof. Let $t \in |X(\xi)|$ be the image point of $|\xi|$ under the natural morphism $\xi \to X(\xi)$.
We note

(1) a) The maximal ideal $\mathfrak{m}_\xi$ of $\mathcal{O}_{X,\xi}$ is the support of t.

b) $\mathcal{O}^+_{X,\xi} = \{f \in \mathcal{O}_{X,\xi} \mid |f(t)| \leq 1\}$.

c) Let $q: X(\xi)_- = \mathrm{Spa}(\mathcal{O}_{X,\xi}, \mathcal{O}^+_{X,\xi}) \to \underline{X}$ be the natural mapping of sets. Then $|X(\xi)| = q^{-1}(|X|)$.

Proof. a) and b) follow from (2.5.10) and (2.5.7).
c) There is a cofinal subcategory E of D_ξ such that, for every $(V, v) \in \mathrm{ob}E$, $|V| = \underline{r}^{-1}(|X|)$ where $r: V \to X$ is the structure morphism of V. This implies $|X(\xi)| = q^{-1}(|X|)$.

i) $\mathcal{O}_{X,\xi}$ is a Tate ring with ring of definition $\mathcal{O}^+_{X,\xi}$. Since $\mathfrak{m}_\xi \subseteq \mathcal{O}^+_{X,\xi}$ (by (1.a, b)) and $\mathfrak{m}_\xi$ is an ideal of $\mathcal{O}_{X,\xi}$, $\mathfrak{m}_\xi$ is contained in every neighbourhood of 0 in $\mathcal{O}_{X,\xi}$. Let $\pi: \mathcal{O}_{X,\xi} \to \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi =: k^\flat$ be the natural mapping, and let $k^+ \subseteq k^\flat$ be the valuation ring of $k^\flat$ corresponding to the valuation t. We equip $k^\flat$ with the valuation topology to k^+. Then $k = (k^\flat, k^+)$ is an affinoid field. Since π is a quotient mapping and $k^+ = \pi(\mathcal{O}^+_{X,\xi})$ (by (1.b)), the mapping between the completions $\pi^\wedge: (\mathcal{O}_{X,\xi}, \mathcal{O}^+_{X,\xi})^\wedge \to (k^\flat, k^+)^\wedge = k^\wedge$ is an isomorphism. By (1.7.3) $(k^\flat)^\wedge$ is the completion of the separable algebraic closure of $k(x)^\wedge$. Then $k^\wedge$ is isomorphic to $\overline{\kappa}(x)^\wedge$. With (1.c) we obtain that $X(\xi)$ is X-isomorphic to $(\mathrm{Spa}\overline{\kappa}(x), p^{-1}(|X|))$.
ii) The image of I in $\mathcal{O}_{X,\xi}$ is contained in the maximal ideal $\mathfrak{m}_\xi$ of $\mathcal{O}_{X,\xi}$ and $\mathfrak{m}_\xi$ is contained in $\mathcal{O}^+_{X,\xi}$ ((1.a, b)). Hence $I \cdot \mathcal{O}_{X,\xi} \subseteq \mathcal{O}^+_{X,\xi}$. Then $(I \cdot \mathcal{O}_{X,\xi})^2 \subseteq I \cdot \mathcal{O}^+_{X,\xi} \subseteq I \cdot \mathcal{O}_{X,\xi}$ which shows that the topology of $\mathcal{O}_{X,\xi}$ is the $I \cdot \mathcal{O}_{X,\xi}$-adic one.
Let $k(B)$ and $k(\xi)$ be the residue fields of B and $\mathcal{O}_{X,\xi}$, and let y' and t' be the valuations of $k(B)$ and $k(\xi)$ induced by the valuations y and t. Since $k(\xi)$ is separably algebraically closed, there exists a A-homomorphism $f: k(B) \to k(\xi)$ which is compatible with the valuations y' and t'. Since $\mathcal{O}_{X,\xi}$ is strictly local, there exists a unique local A-homomorphism $g: B \to \mathcal{O}_{X,\xi}$ which induces f. By (1.7.3) $g(B)$ is dense in $\mathcal{O}_{X,\xi}$. The ring homomorphism $A \to \mathcal{O}_{X,\xi}$ is flat, since $A \to \mathcal{O}_X(U)$ is flat ([Hu$_1$, 1.7.i]) and $\mathcal{O}_X(U) \to \mathcal{O}_{X,\xi}$ is flat ((1.7.6)). Then also $g: B \to \mathcal{O}_{X,\xi}$ is flat. Therefore g is injective and $g^{-1}(I^n \cdot \mathcal{O}_{X,\xi}) = I^n \cdot B$. Hence g is a dense topological embedding. Furthermore, by (1.b) $B^+ = g^{-1}(\mathcal{O}^+_{X,\xi})$. Therefore the completion $g^\wedge: (B, B^+)^\wedge \to (\mathcal{O}_{X,\xi}, \mathcal{O}^+_{X,\xi})^\wedge$ is an isomorphism. Now (1.c) gives that $X(\xi)$ is X-isomorphic to $(\mathrm{Spa}(B, B^+), p^{-1}(|X|))$.

□

Lemma 2.5.14. *Let ξ and ξ' be geometric points of a prepseudo-adic space X.*
i) *The mapping $\mathrm{Hom}_X(X(\xi'), X(\xi)) \to \mathrm{Hom}_X(\xi', X(\xi))$ (induced by the natural morphism $\xi' \to X(\xi')$) is bijective.*
ii) *There exists a X-morphism $X(\xi') \to X(\xi)$ if and only if* supp(ξ') *is a generalization of* supp(ξ) *in $|X|$.*

iii) *Assume that* $\mathrm{supp}(\xi')$ *is a secondary generalization of* $\mathrm{supp}(\xi)$ ((1.1.9)), *and let* $u, v: X(\xi') \rightrightarrows X(\xi)$ *be two* X*-morphisms. Then there exists a* X*-isomorphism* $h: X(\xi') \to X(\xi')$ *with* $u = v \circ h$.

Proof. i) We may assume $\mathrm{Hom}_X(\xi', X(\xi)) \neq \emptyset$. Then by (2.5.10) $\mathrm{supp}(\xi')$ is a generalization of $\mathrm{supp}(\xi)$. We distinguish the cases that $\mathrm{supp}(\xi)$ is an analytic point or a non-analytic point. If $\mathrm{supp}(\xi)$ is analytic then also $\mathrm{supp}(\xi')$ is analytic, and the assertion is obvious by (2.5.13.i). Assume that $\mathrm{supp}(\xi)$ is non-analytic. Let A, A^+, B, B^+, I be as in (2.5.13.ii). Let (K, K^+) and (C, C^+) be the complete affinoid rings with $\underline{\xi'} = \mathrm{Spa}(K, K^+)$ and $X(\xi')_- = \mathrm{Spa}(C, C^+)$. Then K is a separably algebraically closed field, C is a strictly local ring and K is an extension of the residue class field of C. Since $\mathrm{supp}(\xi')$ is a generalization of $\mathrm{supp}(\xi)$, the structure morphisms $\xi' \to X$ and $X(\xi') \to X$ are induced by continuous ring homomorphisms $(A, A^+) \to (K, K^+)$ and $(A, A^+) \to (C, C^+)$. B^+, C^+, K^+ are rings of definition of B, C, K, and IB^+ and IC^+ are ideals of definition of B^+ and C^+, and IK^+ is contained in an ideal of definition of K^+. Therefore the X-morphisms $X(\xi') \to X(\xi)$ correspond to the A-algebra homomorphisms $B \xrightarrow{g} C$ with $g(B^+) \subseteq C^+$, and the X-morphisms $\xi' \to X(\xi)$ correspond to the A-algebra homomorphisms $B \xrightarrow{g} K$ with $g(B^+) \subseteq K^+$. Since $C^+ = K^+ \cap C$ ((2.5.7)) and B is a filtered inductive limit of étale A-algebras, we obtain $\mathrm{Hom}_X(X(\xi'), X(\xi)) \xrightarrow{\sim} \mathrm{Hom}_X(\xi', X(\xi))$.
ii) First we note

(1) Let $p: X(\xi) \to X$ be the structure morphism. Then $\mathrm{im}(|p|)$ is the set of the generalizations of $\mathrm{supp}(\xi)$ in $|X|$.

Proof. Every element of $\mathrm{im}(|p|)$ is a generalization of $\mathrm{supp}(\xi)$ by (2.5.10). Let y be a generalization of $\mathrm{supp}(\xi)$ in $|X|$. We show that $y \in \mathrm{im}(|p|)$. Since $|X(\xi)| = \underline{p}^{-1}(|X|)$ ((2.5.13.)), it suffices to show that $y \in \mathrm{im}(\underline{p})$. We may assume that $\underline{X}$ is affinoid. Let D_ξ be the category of (2.5.9). For every $(V, v) \in \mathrm{ob} D_\xi$, let $p_V: \underline{V} \to \underline{X}$ be the structure morphism. $\underline{p}^{-1}(y) = \varprojlim_{(V,v)\in D_\xi} p_V^{-1}(y)$ (since $X(\xi)_- = \varprojlim_{(V,v)\in D_\xi} \underline{V}$) and every $p_V^{-1}(y)$ is non-empty (since p_V is open by (1.7.8)). Therefore $\underline{p}^{-1}(y) \neq \emptyset$. This proves (1).

If there exists a X-morphism $X(\xi') \to X(\xi)$ then $\mathrm{supp}(\xi')$ is a generalization of $\mathrm{supp}(\xi)$ by (1). Assume that $\mathrm{supp}(\xi')$ is a generalization of $x := \mathrm{supp}(\xi)$. We show $\mathrm{Hom}_X(\xi', X(\xi)) \neq \emptyset$. This is obvious if x is an analytic point of $\underline{X}$ ((2.5.13.i)). Assume that x is a non-analytic point of $\underline{X}$.

We use the notations of (2.5.13.ii). Put $D := \mathcal{O}_{\underline{\xi'}}(\underline{\xi'})$ and $D^+ := \mathcal{O}^+_{\underline{\xi'}}(\underline{\xi'})$. By (1) there exists a $z \in \mathrm{Spa}(B, B^+)$ lying over the valuation $\mathrm{supp}(\xi')$ of A. Let L be the residue class field of B at the prime ideal $\mathrm{supp}(z)$ of B and let $L^+ \subseteq L$ be the valuation ring of L induced by z. Since D is a separably algebraically closed field and L is separably algebraic over the residue class field of A at the prime

ideal $\mathrm{supp}(z) \cap A = \mathrm{supp}(\mathrm{supp}(\xi'))$, there exists a A-homomorphism $f: L \to D$ with $f(L^+) \subseteq D^+$. The composition $(B, B^+) \to (L, L^+) \xrightarrow{f} (D, D^+)$ is continuous. Hence $\mathrm{Hom}_X(\xi', X(\xi)) \neq \emptyset$.
iii) Put $x := \mathrm{supp}(\xi) \in |X|$ and $y := \mathrm{supp}(\xi') \in |X|$. Since $X(\xi) \simeq X(\overline{x})$ and $X(\xi') \simeq X(\overline{y})$, we may assume $\xi = \overline{x}$ and $\xi' = \overline{y}$. Let $u, v: \overline{y} \longrightarrow X(\overline{x})$ be two X-morphisms. We have to show that there exists a X-morphism $h: \overline{y} \to \overline{y}$ with $u = v \circ h$.
By (2.5.13) one has a natural bijection between the set of the X-morphisms $\overline{y} \to X(\overline{x})$ and the set of the $\underline{X}$-morphisms $\mathrm{Spa}\overline{\kappa}(y) \to \mathrm{Spa}\overline{\kappa}(x)$. Let $u_1, v_1: \mathrm{Spa}\overline{\kappa}(y) \longrightarrow \mathrm{Spa}\overline{\kappa}(x)$ be the $\underline{X}$-morphisms corresponding to u, v. We have to show that there exists a $\underline{X}$-morphism $h_1: \mathrm{Spa}\overline{\kappa}(y) \to \mathrm{Spa}\overline{\kappa}(y)$ with $u_1 = v_1 \circ h_1$.
Let $U = \mathrm{Spa}(A, A^+)$ be an open affinoid neighbourhood of x in $\underline{X}$, let k be the residue class field of A at $\mathrm{supp}(x) = \mathrm{supp}(y) \in \mathrm{Spec}\, A$, let $V(x)$ and $V(y)$ be the valuation rings of k corresponding to x and y, let $\overline{k}$ be a separable algebraic closure of k, and let $\overline{V}(x)$ and $\overline{V}(y)$ be valuation rings of $\overline{k}$ which extend $V(x)$ and $V(y)$. The $\underline{X}$-morphisms $u_1, v_1: \mathrm{Spa}\overline{\kappa}(y) \longrightarrow \mathrm{Spa}\overline{\kappa}(x)$ correspond to k-isomorphism $u_2, v_2: \overline{k} \longrightarrow \overline{k}$ with $u_2(\overline{V}(x)) \subseteq \overline{V}(y)$ and $v_2(\overline{V}(x)) \subseteq \overline{V}(y)$. We have to show that there exists a k-isomorphism $h_2: \overline{k} \to \overline{k}$ with $h_2(\overline{V}(y)) = \overline{V}(y)$ and $u_2 = h_2 \circ v_2$. Of course, we have to define $h_2 := u_2 \circ v_2^{-1}$. Then $u_2(\overline{V}(x)) = h_2(v_2(\overline{V}(x)))$. Since $v_2(\overline{V}(x)) \subseteq \overline{V}(y)$, we obtain $u_2(\overline{V}(x)) \subseteq h_2(\overline{V}(y))$. Thus $\overline{V}(y)$ and $h_2(\overline{V}(y))$ are valuation rings of $\overline{k}$ which have the same intersection with k and which contain the valuation ring $u_2(\overline{V}(x))$. This implies $h_2(\overline{V}(y)) = \overline{V}(y)$. □

Lemma 2.5.15. *Let X be a strictly local prepseudo-adic space and ξ a geometric point of X whose support is the closed point of $\underline{X}$. Then, for every sheaf F on X_{et},*

$$\Gamma(X, F) \xrightarrow{\sim} F_\xi.$$

Proof. The assertion follows immediately from the definition of strictly local prepseudo-adic spaces ((2.5.6.i)) and (2.5.4). □

(2.5.16) Let X be a pseudo-adic space. For geometric points ξ, ξ' of X, a specialization morphism $\xi \to \xi'$ is by definition a X-morphism of pseudo-adic spaces $X(\xi) \to X(\xi')$. The geometric points of X together with the specialization morphisms form a category which is denoted by $\mathrm{Pt}(X_{et})$.
For every geometric point ξ of X, we have the stalk functor

$$c_\xi: X_{et}^{\sim} \longrightarrow (\mathrm{sets}), F \mapsto F_\xi.$$

Every specialization morphism $u: \xi \to \xi'$ between geometric points ξ, ξ' of X induces a morphism of functors

$$u^*: c_{\xi'} \longrightarrow c_\xi.$$

Namely, for every sheaf F on $X_{\text{ét}}$, let

$$u^*(F): F_{\xi'} \longrightarrow F_\xi$$

be the mapping which makes the following diagram

$$\begin{array}{ccc} \Gamma(\xi, F) & \xleftarrow{u^*(F)} & \Gamma(\xi', F) \\ \uparrow{\scriptstyle a} & & \uparrow{\scriptstyle b} \\ \Gamma(X(\xi), F) & \xleftarrow{c} & \Gamma(X(\xi'), F) \end{array}$$

commutative where c is induced by the X-morphism $X(\xi) \to X(\xi')$ corresponding to u and a and b are induced by the natural X-morphisms $\xi \to X(\xi)$ and $\xi' \to X(\xi')$. By (2.5.15) a and b are bijective.

A point of the topos $X^{\sim}_{\text{ét}}$ is a morphism of toposes (sets) $\to X^{\sim}_{\text{ét}}$ from the topos (sets) to the topos $X^{\sim}_{\text{ét}}$. Such a morphism is given by a functor $X^{\sim}_{\text{ét}} \to$ (sets) which commutes with inductive limits and finite projective limits. A morphism from a point $f: X^{\sim}_{\text{ét}} \to$ (sets) to a point $f': X^{\sim}_{\text{ét}} \to$ (sets) is a morphism of functors $f' \to f$. The category of points of $X^{\sim}_{\text{ét}}$ is denoted by $\mathrm{Pt}(X^{\sim}_{\text{ét}})$.

Similarly as in [SGA 4, VIII.7.9] one can prove

Proposition 2.5.17. *If X is an adic space then the functor*

$$\begin{array}{ccc} \mathrm{Pt}(X_{\text{ét}}) & \longrightarrow & \mathrm{Pt}(X^{\sim}_{\text{ét}}) \\ \xi & \longmapsto & c_\xi \end{array}$$

is an equivalence to categories.

2.6 The stalks of $R^q f_*$ for quasi-compact morphisms

Proposition 2.6.1. *Let $f: X \to Y$ be a morphism of pseudo-adic spaces which is of weakly finite type and quasi-separated, let ξ be a geometric point of Y and let $Y(\xi)$ be the strict localization of Y at ξ. Let F be a sheaf of sets (resp. groups, resp. abelian groups) on $X_{\text{ét}}$ and let G be the preimage of F on $(X \times_Y Y(\xi))_{\text{ét}}$. Then, for $n = 0$ (resp. $n \in \{0, 1\}$, resp. $n \in \mathbb{N}_0$), there is a natural bijection*

$$(R^n f_* F)_\xi \xrightarrow{\sim} H^n(X \times_Y Y(\xi), G).$$

Important for applications of (2.6.1) is that, for analytic pseudo-adic spaces Y, the strict localization $Y(\xi)$ has a very simple structure ((2.5.13.i)).

A morphism of adic spaces $f: X \to Y$ is locally of algebraically finite type if and only if, for every $x \in X$, there exist open neighbourhoods U and V of x and $f(x)$ in X and Y such that $f(U) \subseteq V$ and U is V-isomorphic to an open subspace of an adic space of the shape $V \times_{\mathrm{Spec}\mathcal{O}_Y(V)} L$ where L is a scheme of finite type over $\mathrm{Spec}\mathcal{O}_Y(V)$.

1.3 Separated morphisms and proper morphisms

Definition 1.3.1. (Separated morphisms) A morphism of adic spaces $f: X \to Y$ is called *separated* if f is locally of weakly finite type and $\Delta(X)$ is closed in $X \times_Y X$ where $\Delta: X \to X \times_Y X$ is the diagonal morphism of f.

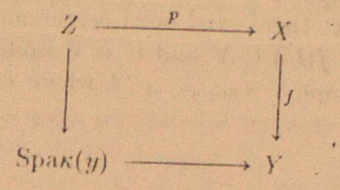
Definition 1.3.2. (Proper morphisms) Let $f: X \to Y$ be a morphism of adic spaces.
i) f is called *universally closed* if f is locally of weakly finite type and, for every adic morphism of adic spaces $Y' \to Y$, the projection $X \times_Y Y' \to Y'$ is closed.
ii) f is called *proper* if f is of $^+$weakly finite type, separated and universally closed.

Definition 1.3.3. (Partially proper morphisms) Let $f: X \to Y$ be a morphism of adic spaces.
i) f is called *specializing at a point* $x \in X$ if, for every specialization y' of $f(x)$ in Y, there exists a specialization x' of x in X with $y' = f(x')$. f is called *universally specializing at a point* $x \in X$ if f is locally of weakly finite type and, for every adic morphism of adic spaces $Y' \to Y$ and every point x' of $X \times_Y Y'$ lying over x, the projection $X \times_Y Y' \to Y'$ is specializing at x'. f is called *specializing* if f is specializing at every point of X, and f is called *universally specializing* if f is universally specializing at every point of X. Similarly we define *generalizing* and *universally generalizing* morphisms of adic spaces.
ii) f is called *partially proper* if f is locally of $^+$weakly finite type, separated and universally specializing.

In (1.3.18) and (1.3.19) at the end of this paragraph we will compare proper and partially proper morphisms of adic spaces with proper morphisms of formal schemes and with proper morphisms of rigid analytic varieties.

Lemma 1.3.4. *For a morphism of adic spaces $f: X \to Y$ the following conditions are equivalent:*
i) f is proper.
ii) f is partially proper and quasi-compact.

Proof. A morphism of adic spaces $g: U \to V$ is specializing at a point $u \in U$ iff $g(\overline{\{u\}})$ is closed in V.
Hence (i) implies (ii). In order to show that (ii) implies (i), we show that every quasi-compact and specializing morphism of adic spaces $g: U \to V$ is closed. Let



Let G be the set of the secondary generalizations of y in Y. Then p induces a homeomorphism from Z onto the topological subspace $f^{-1}(G)$ of X, and, for every $z \in Z$, $k(p(z))$ is dense in $k(z)$.

Definition 1.2.5. i) A ring homomorphism between f-adic rings $f: A \to B$ (A and B are not necessarily complete) is called *of algebraically finite type* if f is adic and if there exist rings of definition A_0 and B_0 of A and B such that $f: A \to B$ and $f|A_0: A_0 \to B_0$ are ring homomorphisms of finite type.
ii) A ring homomorphism between affinoid rings $f: A \to B$ is called *of algebraically finite type* if the ring homomorphism of f-adic rings $f: A^b \to B^b$ is of algebraically finite type and if there exists an open subring D of B^b such that B^+ is the integral closure of D in B^b and the ring homomorphism of f-adic rings $f|A^+: A^+ \to D$ is of algebraically finite type.
iii) A morphism $f: X \to Y$ between adic spaces is called *locally of algebraically finite type* if, for every $x \in X$, there exist an open affinoid neighbourhood U of x in X, an open affinoid subspace V of Y and a homomorphism of algebraically finite type between affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to A$ such that $f(U) \subseteq V$ and U is V-isomorphic to $\mathrm{Spa}A$.

Remark 1.2.6. i) If $A \to B$ is a ring homomorphism of algebraically finite type between f-adic (resp. affinoid) rings then $A \to B$ is of topologically finite type.
ii) If $A \to B$ is a ring homomorphism of algebraically finite type between f-adic rings then there exists a finite subset L of B such that, for every ring of definition A_0 of A, the subring $A_0[L]$ of B is a ring of definition of B.
iii) If $A \to B$ is a ring homomorphism of algebraically finite type between affinoid rings then there exists a finite subset L of B^b such that B^+ is the integral closure of $A^+[L]$ in B^b and, for every ring of definition A_0 of A^b, $A_0[L]$ is a ring of definition of B^b.

(1.2.7) Let t denote the forgetful functor from the category of adic spaces to the category of locally ringed spaces which assigns to an adic space $(X, \mathcal{O}_X, (v_x \mid x \in X))$ the locally ringed space $(X, \mathcal{O}_X)$. If X is an adic space and S a locally ringed space then, for a morphism of locally ringed spaces $f: t(X) \to S$, we very often write $f: X \to S$.

For morphisms of locally ringed spaces $g: T \to S$ and $f: X \to S$ where T, S are schemes, X is an adic space and g is locally of finite type, we have an adic space $X \times_S T$ over X ([Hu, 3.8]).

b) *For every $(x, A) \in X_v$ and every $y \in Y$, there exists at most one center $z \in X$ of (x, A) with $y = f(z)$.*

Proof. It is easily seen that f is quasi-separated if and only if the diagonal morphism $\Delta: X \to X \times_Y X$ is quasi-compact. Let $p_1, p_2: X \times_Y X \rightrightarrows X$ be the projections.
a) $\Longrightarrow$ b): Let $x_1, x_2 \in X$ be centers of some $A \in X_v$ with $f(x_1) = f(x_2) =: y$. We choose open affinoid neighbourhoods U_1, U_2, V of x_1, x_2, y on X and Y such that $f(U_i) \subseteq V$ and $\mathcal{O}_X(U_i)$ is of topologically finite type over $\mathcal{O}_Y(V)$ $(i = 1, 2)$. By (1.3.6.ii) $\Delta_v(A)$ has a center z on $U_1 \times_V U_2 \subseteq X \times_Y X$. Then $p_i(z) \in U_i$ is a center of A on U_i. Hence by (1.3.6.ii) $x_i = p_i(z)$. Since $\Delta(X)$ is closed in $X \times_Y X$, we have $z \in \Delta(X)$. Hence $x_1 = x_2$.
b) $\Longrightarrow$ a): $\Delta(X)$ is pro-constructible in $X \times_Y X$, since Δ is quasi-compact. Thus, in order to show that $\Delta(X)$ is closed in $X \times_Y X$, it suffices to show that $\Delta(X)$ is closed under specializations in $X \times_Y X$. Let x be a point of $\Delta(X)$ and $y \in X \times_Y X$ a specialization of x. By (1.3.6.i) there exists a $A \in (X \times_Y X)_v$ such that x is the support of A and y is a center of A. Then $p_1(y)$ and $p_2(y)$ are centers of $(p_1)_v(A) = (p_2)_v(A) \in X_v$ with $f(p_1(y)) = f(p_2(y))$. Hence $p_1(y) = p_2(y)$. We choose open affinoid subspaces U, V of X, Y such that $p_1(y) = p_2(y) \in U$, $f(U) \subseteq V$ and $\mathcal{O}_X(U)$ is of topologically finite type over $\mathcal{O}_Y(V)$. Then $y \in p_1^{-1}(U) \cap p_2^{-1}(U) = U \times_V U$. Since $\Delta(X) \cap (U \times_V U)$ is closed in $U \times_V U$, we obtain $y \in \Delta(X)$. □

Proposition 1.3.8. *Let $f: X \to Y$ be a morphism locally of weakly finite type between analytic adic spaces, and let $x_0 \in X$ be a point. The following conditions are equivalent:*

a) *f is universally specializing at x_0.*

b) *For every $(x_0, A) \in X_v$ and every center $y \in Y$ of (x_0, A) on Y, there exists a center $x \in X$ of (x_0, A) on X with $y = f(x)$.*

Proof. a) $\Longrightarrow$ b): Put $x := x_0$. Let (x, A) be a valuation ring of X and let $y \in Y$ be a center of $f_v((x, A))$. We show that there exists a center $z \in X$ of (x, A) with $y = f(z)$. As defined in (1.1.7) $k(x)$ carries a topology. Then $(k(x), k(x)^+)$ and $(k(x), A)$ are affinoid rings. The identity $k(x) \to k(x)$ induces a morphism of adic spaces $f'': X'' := \mathrm{Spa}(k(x), k(x)^+) \to \mathrm{Spa}(k(x), A) =: Y'$. Let U and V be open affinoid subspaces of X and Y with $x \in U$, $y \in V$ and $f(U) \subseteq V$. The natural ring homomorphisms $\mathcal{O}_X(U) \to k(x)$ and $\mathcal{O}_Y(V) \to k(x)$ induce morphisms $g'': X'' \to X$ and $g: Y' \to Y$. The diagram

$$(*) \qquad \begin{array}{ccc} X'' & \xrightarrow{g''} & X \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

L be a closed subset of U. Since g is quasi-compact, $g(L)$ is pro-constructible in V. Hence $g(L)$ is closed in V iff $g(L)$ is closed under specializations in V. But the latter is obviously satisfied, since L is closed under specializations in U and g is specializing. □

There is a valuative criterion for separatedness and properness of morphisms of adic spaces ([Hu$_4$]). For simplicity we explain here this criterion only for morphisms between analytic adic spaces.

Definition 1.3.5. Let X be an analytic adic space. A *valuation ring of* X is a pair (x, A), where x is a point of X and A is a valuation ring of the residue class field $k(x)$ of x with $A \subseteq k(x)^+$. (Often we write only A instead of (x, A).) x is called the *support* of (x, A). A point $y \in X$ is called a *center* of (x, A) if y is a specialization of x with $k(y)^+ = i^{-1}(A)$ where $i: k(y) \to k(x)$ is the natural ring homomorphism ((1.1.10.iii)).

Lemma 1.3.6. i) *Let X be an analytic adic space and let x, y be points of X with $x \rightsquigarrow y$. Then there exists a unique valuation ring A of X such that x is the support of A and y is a center of A.*
ii) *Let (x, A) be a valuation ring of an analytic affinoid adic space X. Then (x, A) has at most one center on X. (x, A) has a center on X if and only if $f(x) \in A$ for every $f \in \mathcal{O}_X^+(X)$.*

Proof. i) follows from (1.1.10.iii).
ii) Let $y, z \in X$ be centers of (x, A). Let $\varphi: \mathcal{O}_X(X) \to \mathcal{O}_{X,y}$ and $\psi: \mathcal{O}_X(X) \to \mathcal{O}_{X,z}$ be the natural ring homomorphisms. Then $\mathrm{Spv}(\varphi)(v_y) = \mathrm{Spv}(\psi)(v_z)$. Hence $y = z$.

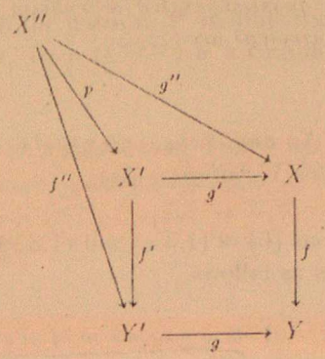
For every $f \in \mathcal{O}_X^+(X)$ and $y \in X$, we have $f(y) \in k(y)^+$. Hence if (x, A) has a center on X then $f(x) \in A$ for every $f \in \mathcal{O}_X^+(X)$.
Now we assume that $f(x) \in A$ for every $f \in \mathcal{O}_X^+(X)$. Then the valuation y of the ring $\mathcal{O}_X(X)$ induced by A via the natural ring homomorphism $\mathcal{O}_X(X) \to k(x)$ is an element of $X = \mathrm{Spa}(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$. Since $A \subseteq k(x)^+$, y is a specialization of x. Let $i: k(y) \to k(x)$ be the natural ring homomorphism, and let $k \subseteq k(y)$ be the subfield of $k(y)$ which is generated by the image of $\mathcal{O}_X(X)$ in $k(y)$. Then $i^{-1}(A) \cap k = k(y)^+ \cap k$. Since k is dense in $k(y)$, we obtain $i^{-1}(A) = k(y)^+$. □

For an analytic adic space X, let X_v denote the set of all valuation rings of X. Every morphism of analytic adic spaces $f: X \to Y$ induces a mapping $f_v: X_v \to Y_v$, $(x, A) \mapsto (f(x), A \cap k(f(x)))$. If $p \in X$ is a center of $(x, A) \in X_v$ then $f(p) \in Y$ is a center of $f_v((x, A)) \in Y_v$. (If there is no ambiguity on f then we often call $y \in Y$ a center of $(x, A) \in X_v$ on Y if y is a center of $f_v((x, A))$.)

Proposition 1.3.7. (Valuative criterion for separatedness) *Let $f: X \to Y$ be a quasi-separated morphism locally of weakly finite type between analytic adic spaces. The following conditions are equivalent:*

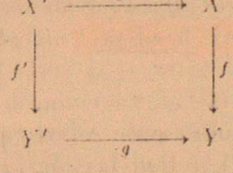
a) *f is separated.*

commutes. Put $X' := X \times_Y Y'$, and let $f' : X' \to Y'$ and $g' : X' \to X$ be the projections. Then $(\star)$ induces a commutative diagram



Let $x'' \in X''$ be the closed point of X'' and let $y' \in Y'$ be the closed point of Y'. Then $g''(x'') = x$ and $g(y') = y$. We consider the point $x' := p(x'') \in X'$. Then $g'(x') = x$ and y' is a specialization of $f'(x')$. Since f' is specializing at x', there exists a specialization $z' \in X'$ of x' with $f'(z') = y'$. Then $z := g'(z') \in X$ is a specialization of x with $y = f(z)$. Obviously z is a center of (x, A).

b) $\Longrightarrow$ a): Let



be a cartesian diagram of adic spaces, and let $x'_0 \in X'$ be a point of X' with $x_0 = g'(x'_0)$. We show that f' is specializing at x'_0. Let $y' \in Y'$ be a specialization of $y'_0 := f'(x'_0) \in Y'$. We choose a valuation ring A' of $k(x'_0)$ such that $A \subseteq k(x'_0)^+$ and $i^{-1}(A' \cap k(y'_0)) = k(y')^+$ where $i : k(y') \to k(y'_0)$ is the natural ring homomorphism. Then $(x'_0, A') \in X'_v$ and y' is a center of $f'_v((x'_0, A'))$. Put $y = g(y')$ and $(x_0, A) := g'_v((x'_0, A')) \in X_v$. Then y is a center of $f_v((x_0, A))$. Hence (x_0, A) has a center $x \in X$ with $y = f(x)$. We choose open affinoid subspaces U, V, V' of X, Y, Y' such that $x \in U$, $y' \in V'$, $f(U) \subseteq V$, $g(V') \subseteq V$, and $\mathcal{O}_X(U)$ is of topologically finite type over $\mathcal{O}_Y(V)$. We have $f'(x'_0) \in V'$ and $g'(x'_0) \in U$. Hence $x'_0 \in U \times_V V'$. By (1.3.6.ii) (x'_0, A') has a center x' on $U \times_V V'$. Then $f'(x')$ is a center of $f'_v((x'_0, A'))$ on V'. Hence by (1.3.6.ii) $y' = f'(x')$. □

Corollary 1.3.9. (Valuative criterion for properness and partially properness)
i) Let $f : X \to Y$ be a quasi separated morphism locally of $^+$weakly finite type between analytic adic spaces. Equivalent are

a) *f is partially proper*

b) *For every $(x, A) \in X_v$ and every center y of (x, A) on Y, there exists a unique center z of (x, A) on X with $y = f(z)$.*

ii) *Let $f: X \to Y$ be a quasi-separated morphism of $^+$weakly finite type between analytic adic spaces. Equivalent are*

a) *f is proper.*

b) *For every $(x, A) \in X_v$ and every center y of (x, A) on Y, there exists a unique center z of (x, A) on X with $y = f(z)$.*

Obviously the conditions (b) of (1.3.7) and (1.3.9) can be formulated in terms of morphisms of adic spaces as follows:

Lemma 1.3.10. *Let $f: X \to Y$ be a morphism of analytic adic spaces. The condition* (b) *of (1.3.7) (resp. (1.3.9)) is equivalent to the following condition: For every commutative diagram of adic spaces*

$$\begin{array}{ccc} S & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[g]{} & Y \end{array}$$

where S and T are spectra of affinoid fields such that $\mathcal{O}_T(T) \to \mathcal{O}_S(S)$ is bijective, there exists at most one (resp. a unique) morphism $p: T \to X$ with $g = f \circ p$ and $g' = p \circ f'$.

(1.3.11) For every analytic adic space X, we equip X_v with the topology which is generated by the sets $D(U, f) := \{(x, A) \in X_v \mid x \in U \text{ and } f(x) \in A\}$ where $U \subseteq X$ is open and $f \in \mathcal{O}_X(U)$.

Lemma 1.3.12. *Let X be a quasi-compact quasi-separated analytic adic space. Then X_v is a spectral space. For every quasi-compact open subset U of X and every $f \in \mathcal{O}_X(U)$, the subset $D(U, f)$ of X_v is quasi-compact.*

Proof. In [Hu$_1$] we proved that X is a spectral space. With the same arguments one can prove that X_v is a spectral space. We give another proof which uses that X and abstract Riemann surfaces of fields are spectral spaces. Let X_v^c denote the set X_v equipped with the topology which is generated by the sets $D(U, f)$, $X_v - D(U, f)$ where U is a quasi-compact open subset of X and $f \in \mathcal{O}_X(U)$. In order to show (1.3.12) it suffices to show that X_v^c is quasi-compact ([Ho, Prop. 7]). But this follows from:

(1) There exists a topological space X^c and a mapping $p: X_v^c \to X^c$ such that p is continuous and closed, X^c is quasi-compact and, for every $x \in X^c$, $p^{-1}(x)$ is quasi-compact.

(1.4.2) A ring homomorphism of complete affinoid rings $f: A \to B$ is called *finite* if f is of topologically finite type and the ring homomorphisms $A^\flat \to B^\flat$ and $A^+ \to B^+$ are integral. Then $B^\flat$ is finite over $A^\flat$, the topology of $B^\flat$ is the natural $A^\flat$-module topology ([Hu, §2]) and B^+ is the integral closure of A^+ in $B^\flat$. Conversely, if C is a finite $A^\flat$-algebra equipped with the natural $A^\flat$-module topology and if C^+ denotes the integral closure of A^+ in C then (C, C^+) is a complete affinoid ring which is finite over A ([Hu$_4$, 3.12.10]).

Lemma 1.4.3. *Let A and B be affinoid rings which are complete and Tate, and let $A \to B$ be a morphism of topologically finite type. Let $t_1, \ldots, t_n$ be elements of B^+ such that the A-morphism $p: A\langle T_1, \ldots, T_n\rangle \to B, T_i \mapsto t_i$ for $i = 1, \ldots, n$ is a quotient mapping* ([Hu, 3.5.iii]).
Equivalent are

a) *B is finite over A.*

b) *For every $i \in \{1, \ldots, n\}$, there exists a monic polynomial $p_i \in A^+[X]$ with $p_i(t_i) \in (B^\flat)^{\circ\circ}$.*

Proof. If B is finite over A then B^+ is integral over A^+ and hence b) is satisfied. Now assume b), and we show that B is finite over A. Let A_0 be a ring of definition of A^+ which contains all coefficients of the p_i, and let I be an ideal of definition of A_0. The A_0-algebra $B_0 := p(A_0\langle T_1, \ldots, T_n\rangle)$ is a ring of definition of B^+ with $t_i \in B_0$ for $i = 1, \ldots, n$ and $A_0[t_1, \ldots, t_n]$ is dense in B_0. Since $p_i(t_i) \in (B^\flat)^{\circ\circ}$, there exists a $m \in \mathbb{N}$ with $p_i(t_i)^m \in IB_0$ for $i = 1, \ldots, n$. Replacing p_i by p_i^m, we may assume that $p_i(t_i) \in IB_0$ for $i = 1, \ldots, n$. Let $\bar{t}_i$ be the image of t_i in B_0/IB_0. Then $B_0/IB_0 = A_0[\bar{t}_1, \ldots, \bar{t}_n]$ and every $\bar{t}_i$ is integral over A_0. Hence B_0/IB_0 is finite over A_0. This implies that B_0 is finite over A_0 ([M, 28.P.Lemma]). Hence B_0 is integral over A_0. Since $A\langle T_1, \ldots, T_n\rangle^+$ is the integral closure of $A^+ A_0\langle T_1, \ldots, T_n\rangle$ in $A\langle T_1, \ldots, T_n\rangle^\flat$, we obtain that B^+ is integral over A^+. Then also $B^\flat = A^\flat \otimes_{A^+} B^+$ is integral over $A^\flat$. □

(1.4.4) A morphism of adic spaces $f: X \to Y$ is called *finite* if, for every $y \in Y$, there exists an open affinoid neighbourhood V of y in Y such that $U := f^{-1}(V)$ is affinoid and the ring homomorphism of affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is finite. Then, for every open affinoid subspace V of Y, $U := f^{-1}(V)$ is affinoid and $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is finite over $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$ ([Hu$_4$, 3.6.20, 3.12.12], [Hu$_2$]).

Lemma 1.4.5. i) *Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

(a) $f: X \to Y$ is separated

(b) $r(f): r(X) \to r(Y)$ is separated.

iii) We consider the conditions

(a) $f: X \to Y$ is partially proper

(b) $r(f): r(X) \to r(Y)$ is partially proper.

Then (a) implies (b) and if the valuation ring k° is discrete then (b) implies (a). (The proof of the first statement is easy (see proof of (1.5.9)). The second statement follows from [L_1] and (1.3.18.iii).)

iv) With (iii) we obtain that if we consider the statements

(a) $f: X \to Y$ is proper

(b) $r(f): r(X) \to r(Y)$ is proper

then (a) implies (b) and if k° is discrete then (b) implies (a).

1.4 Finite morphisms

(1.4.1) If $A = (A^\circ, A^+)$ is an affinoid ring and J an ideal of A° then A/J denotes the affinoid ring $(A^\circ/J, (A^+/J \cap A^+)^c)$ where A°/J carries the quotient topology of A° and $(A^+/J \cap A^+)^c$ is the integral closure of $A^+/J \cap A^+$ in A°/J.

Let $X = (X, \mathcal{O}_X, (v_x \mid x \in X))$ be an adic space, and let $I \subseteq \mathcal{O}_X$ be a quasi-coherent $\mathcal{O}_X$-module. Put $Y := \{x \in X \mid I_x \neq \mathcal{O}_{X,x}\}$, $\mathcal{O}_Y := (\mathcal{O}_X/I) \mid Y$, and, for every $y \in Y$, let v'_y denote the valuation of $\mathcal{O}_{Y,y}$ induced by v_y. There is a unique ring topology on the sheaf $\mathcal{O}_Y$ such that, for every open affinoid subspace U of X, the mapping $\mathcal{O}_X(U) \to \mathcal{O}_Y(Y \cap U)$ is continuous and open. The triple $Y = (Y, \mathcal{O}_Y, (v'_y \mid y \in Y))$ is an adic space ([Hu_4, 3.6], [Hu_2]). If $X = \mathrm{Spa}A$ and $J = \Gamma(X, I)$ then $Y = \mathrm{Spa}A/J$. The adic spaces of the form $(Y, \mathcal{O}_Y, (v'_y \mid y \in Y))$ are called the *closed adic subspaces* of X.

A morphism of adic spaces $f: Z \to X$ is called a closed (resp. open, resp. locally closed) embedding if f has a factorization $f = i \circ g$ where $i: Y \to X$ is the natural morphism from a closed (resp. open, resp. locally closed) adic subspace Y of X to X and $g: Z \to Y$ is an isomorphism.

Let k be a complete non-archimedean field. For every Tate ring A of topologically finite type over k and every ideal J of A, we have $(A, A^\circ)/J = (A/J, (A/J)^\circ)$ ([BGR, 6.3.4 Prop 1]). Therefore, a morphism $f: X \to Y$ of rigid analytic varieties over k is a closed embedding if and only if the associated morphism of adic spaces $r(f): r(X) \to r(Y)$ is a closed embedding.

Proposition 5.8. *Let $f : Z \to X$ be a map of perfectoid spaces, where X is affinoid.*

(i) *If f is strongly Zariski closed, then f is Zariski closed, and in particular a closed immersion.*

(ii) *If f is Zariski closed, then Z is affinoid.*

(iii) *If X is of characteristic p and f is Zariski closed, then f is strongly Zariski closed.*

Proof. Cf. [Sch15, Section II.2]. □

A general class of immersions is given by diagonal morphisms.

Proposition 5.9. *Let $f : Y \to X$ be any map of perfectoid spaces. Then the diagonal morphism*

$$\Delta_f : Y \to Y \times_X Y$$

is an immersion.

Proof. Clearly, Δ_f is an injection. Thus, it is enough to show that $|\Delta_f|$ identifies $|Y|$ with a locally closed subspace of $|Y \times_X Y|$. This statement can be checked locally on X and Y; thus, we can assume that $X = \mathrm{Spa}(R, R^+)$ and $Y = \mathrm{Spa}(S, S^+)$ are affinoid. But then Δ_f is strongly Zariski closed, as $S \widehat{\otimes}_R S \to S$ is surjective. By Proposition 5.8, it follows that Δ_f is a closed immersion. □

Definition 5.10. *A map $f : Y \to X$ of perfectoid spaces is* separated *if Δ_f is a closed immersion.*

Proposition 5.11. *Let $f : Y \to X$ be a map of perfectoid spaces. The following conditions are equivalent.*

(i) *The map f is separated.*

(ii) *The map $|\Delta_f| : |Y| \to |Y \times_X Y|$ is a closed immersion.*

(iii) *The map f is quasiseparated (i.e., $|f| : |Y| \to |X|$ is quasiseparated), and for all perfectoid fields K with ring of integers $\mathcal{O}_K$ and an open and bounded valuation subring $K^+ \subset K$ and any commutative diagram*

$$\begin{array}{ccc} \mathrm{Spa}(K, \mathcal{O}_K) & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spa}(K, K^+) & \longrightarrow & X \end{array},$$

there exists at most one dotted arrow making the diagram commute.

Proof. The equivalence of (i) and (ii) follows from Proposition 5.9. Next, we check that (ii) implies (iii). If $|\Delta_f|$ is a closed immersion, then it is in particular quasicompact; this implies that f is quasiseparated. Now assume that there are two dotted arrows making the diagram commute. These define a point of $z \in (Y \times_X Y)(K, K^+)$ such that $z|_{(K,\mathcal{O}_K)} \in \Delta_f(Y)(K, \mathcal{O}_K)$. But if $|\Delta_f| : |Y| \to |Y \times_X Y|$ is a closed immersion, then the map $z : \mathrm{Spa}(K, K^+) \to Y \times_X Y$ maps into $|\Delta_f|(|Y|)$ if this is true for $z|_{(K,\mathcal{O}_K)}$, as $\mathrm{Spa}(K, \mathcal{O}_K) \subset \mathrm{Spa}(K, K^+)$ is dense. As Δ_f is an immersion, this implies that z maps $\mathrm{Spa}(K, K^+)$ into $\Delta_f(Y)$, so that the dotted arrows agree.

Finally, (iii) implies (ii): The conditions in (iii) imply that the map $|\Delta_f| : |Y| \to |Y \times_X Y|$ is a quasicompact locally closed immersion of locally spectral spaces, which is moreover specializing. We claim that this implies that $|\Delta_f|$ is a closed immersion. By quasicompactness, the image of $|\Delta_f|$ is pro-constructible, thus the closure is given by the set of specializations; but the image is already closed under specializations. □

[Hub96] p. 80.– §1.7.

[Hub96]

§1.7., p. 80.

§2.55 p. 87

6. Étale morphisms

In this section, we recall briefly some facts about étale morphisms of perfectoid spaces. We start with the almost purity theorem.

Thm. 13.2 Étale „Morphisms of schemes"

Theorem 6.1 ([Fal02],[KL15, Theorem 3.6.21, Theorem 5.5.9],[Sch12, Theorem 5.25, Theorem 7.9 (iii)]). *Let R be a perfectoid Tate ring with tilt $R^\flat$.*

(i) *For any finite étale R-algebra S, S is perfectoid.*

(ii) *Tilting induces an equivalence*

$$\{\text{Finite étale } R\text{-algebras}\} \to \{\text{Finite étale } R^\flat\text{-algebras}\}$$
$$S \mapsto S^\flat$$

Lemma 12.3 „Morphisms of sheaves"

(iii) *For any finite étale R-algebras S, S° is almost finite étale over R°.*

Object not defined

Definition 6.2. *Let $f : Y \to X$ be a morphism of perfectoid spaces.*

(i) *The morphism f is finite étale if for all open affinoid perfectoid $\mathrm{Spa}(R, R^+) = U \subset X$, the preimage $V = f^{-1}(U) = \mathrm{Spa}(S, S^+) \subset Y$ is affinoid perfectoid, $R \to S$ is a finite étale morphism of rings, and S^+ is the integral closure of R^+ in S.*

(ii) *The morphism f is étale if for all $y \in Y$, there are open subsets $V \subset Y$, $U \subset X$ such that $y \in Y$, $f(V) \subset U$, and there is a factorization*

$$\begin{array}{ccc} V & \hookrightarrow & W \\ & \searrow^{f|_V} & \downarrow \\ & & U, \end{array}$$

where $V \hookrightarrow W$ is an open immersion, and $W \to U$ is finite étale.

affinoid perfectoid

We refer to [Sch12, Section 7] for an extensive discussion. In particular, it is proved there that a morphism is finite étale if and only if there is a cover by open affinoid perfectoid $U = \mathrm{Spa}(R, R^+) \subset X$ such that their preimages $V = \mathrm{Spa}(S, S^+) \subset Y$ are affinoid perfectoid, $R \to S$ is finite étale, and S^+ is the integral closure of R^+ in S. Moreover, composites, base changes, and maps between (finite) étale maps are (finite) étale.

By $X_{\text{fét}}$, resp. $X_{\text{ét}}$, we denote the categories of finite étale, resp. étale, perfectoid spaces over X. If $X = \mathrm{Spa}(R, R^+)$ is affinoid, then $X_{\text{fét}} \cong R^{\mathrm{op}}_{\text{fét}}$.

Theorem 6.3 ([Sch12], [KL15]). *Let X be a perfectoid space with tilt $X^\flat$. Then tilting induces an equivalence $X_{\text{ét}} \cong X^\flat_{\text{ét}}$.*

If $X = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid, then the R^+-module $H^i(X_{\text{ét}}, \mathcal{O}_X^+)$ is almost zero for $i > 0$, and $H^0(X_{\text{ét}}, \mathcal{O}_X^+) = R^+$.

We will need a result about étale morphisms to an inverse limit of spaces.

Proposition 6.4. *Let $X_i = \mathrm{Spa}(R_i, R_i^+)$, $i \in I$, be a cofiltered inverse system of affinoid perfectoid spaces for some κ-small index category I, where $\lambda < \kappa$. Assume that all X_i are κ'-small for some strong limit cardinal $\kappa' < \kappa$. Then the inverse limit $X = \mathrm{Spa}(R, R^+)$ is still κ-small, where R^+ is the ϖ-adic completion of $\varinjlim_i R_i^+$, and $R = R^+[\frac{1}{\varpi}]$. Here, $\varpi \in R_i^+$ denotes any compatible choice of pseudouniformizers for large i. Moreover, one has the following results.*

Proof. We may assume that $X = \mathrm{Spa}A$ and $Y = \mathrm{Spa}B$ are affinoid. We consider X and Y as topological subspaces of the valuation spectra $\mathrm{Spv}A^\flat$ and $\mathrm{Spv}B^\flat$. All specializations in X and Y are secondary specializations. Then (i) is obvious. For the proof of (ii) observe that the valuations $x \in X$ are microbial. (iv) follows from (ii). The proof of (vi) is easy (cf. (1.1.14)). (v) follows from (vi), (iv), (i).
iii) For rational subsets U and V of X with $U \subseteq V$, $\mathcal{O}_X(U)$ is flat over $\mathcal{O}_X(V)$ ([Hu$_1$, 1.7.i], and (II.1.iv) in the proof of [Hu, 2.5]). Hence j is flat. Since, for every rational subset U of X with $x, y \in U$, y is a secondary specialization of x in $\mathrm{Spv}\mathcal{O}_X(U)$, j is a local ring homomorphism. □

(1.1.11) Let k be a complete non-archimedean field. In [Hu, 4.3] there is constructed a functor $r = r_k$ from the category of rigid analytic varieties over $\mathrm{Sp}k$ ([BGR, 9.3.1]) to the category of adic spaces over $\mathrm{Spa}(k, k^\circ)$. This functor is characterized by the following properties ([Hu$_4$, 3.4.17])

(a) If $X = \mathrm{Sp}A$ is an affinoid rigid analytic variety then $r(X) = \mathrm{Spa}(A, A^\circ)$. If $f: \mathrm{Sp}A \to \mathrm{Sp}B$ is a morphism of affinoid rigid analytic varieties given by a k-algebra homomorphism $\varphi: B \to A$ then $r(f): \mathrm{Spa}(A, A^\circ) \to \mathrm{Spa}(B, B^\circ)$ is the morphism of adic spaces which is induced by the homomorphism $\varphi: (B, B^\circ) \to (A, A^\circ)$.

(b) If $f: X \to Y$ is an open embedding of rigid analytic varieties then $r(f): r(X) \to r(Y)$ is an open embedding of adic spaces.

(c) A family $(X_i)_{i \in I}$ of admissible open subsets of a rigid analytic variety X is an admissible covering of X if and only if $r(X) = \bigcup_{i \in I} r(X_i)$ (According to (b) we consider $r(X_i)$ as an open subset of $r(X)$).

(d) r is fully faithful.

[Huber 94]
Prop. 4.5, p. 543

For every rigid analytic variety X over $\mathrm{Sp}k$, the adic space $r(X)$ is locally of finite type over $\mathrm{Spa}(k, k^\circ)$ (for morphisms of finite type between adic spaces, cf. section (1.2)). r induces an equivalence from the category of quasi-separated rigid analytic varieties over $\mathrm{Sp}k$ to the category of adic spaces over $\mathrm{Spa}(k, k^\circ)$ whose structure morphisms are locally of finite type and quasi-separated. More general, for every rigid analytic variety X, r induces an equivalence from the category of rigid analytic varieties over X whose structure morphisms are quasi separated to the category of adic spaces over $r(X)$ whose structure morphisms are locally of finite type and quasi-separated.

In virtue of (b) above we consider, for an open subvariety U of a rigid analytic variety X, $r(U)$ as an open adic subspace of $r(X)$.

For every rigid analytic variety X over $\mathrm{Sp}k$, there is a natural bijection

$$e: X \xrightarrow{\sim} \{x \in r(X) \mid k(x) \text{ is finite over } k\},$$

namely if x is a point of X and $i: \mathrm{Sp}k(x) \to X$ is the natural morphism then $e(x) \in r(X)$ is the image point of the mapping $r(i): \mathrm{Spa}(k(x), k(x)^\circ) \to r(X)$. We consider X as a subset of $r(X)$ via the mapping e. For every admissible open subset U of X, we have $r(U) \cap X = U$. Hence $U \mapsto r(U)$ gives an injection φ from the set of all admissible open subsets of X into the set of all open subsets of $r(X)$. Of course, φ restricts to an injection ψ from the set of all quasi-compact admissible open subsets of X into the set of all quasi-compact open subsets of $r(X)$. If X is quasi-separated then ψ is a bijection. (Under this bijection the open affinoid subspaces of X correspond to the open affinoid subspaces of $r(X)$). Hence, for every rigid analytic variety X, a constructible subset T of the locally spectral space $r(X)$ (cf. (1.1.13)) is uniquely determined by its intersection $T \cap X$.

The functor r commutes with fibre products.

For every rigid analytic variety X, the morphism of sites

$$\begin{array}{ccc} r(X) & \longrightarrow & X \\ r(U) & \longleftarrow & U \end{array}$$

induces an equivalence of toposes

$$r(X)^\sim \xrightarrow{\sim} X^\sim.$$

(1.1.12) In [BL, 5.1] there is defined, for a noetherian admissible formal scheme S, the category (Rig/S) of rigid S-spaces. (Rig/S) is the localization of the category $(FSch/S)$ of admissible formal S-schemes with respect to the class of admissible formal blowing-ups.
In order to explain the relation between the category (Rig/S) and the category of adic spaces, let us consider the functor t from the category of locally noetherian formal schemes to the category (A) of adic spaces as defined in [Hu, 4.1]. (If $X = \mathrm{Spf}A$ is an affine noetherian formal scheme then $t(X) = \mathrm{Spa}(A, A)$.) Let

$$t_a: (FSch/S) \longrightarrow (A)$$

be the functor which assigns to an admissible formal S-scheme X the open subspace $t(X)_a$ of analytic points of the adic space $t(X)$. t_a transforms admissible blowing-ups to isomorphisms. Hence t_a factors through a functor

$$t'_a: (Rig/S) \longrightarrow (A).$$

t'_a gives an equivalence from the category (Rig/S) to the category of adic spaces over $t(S)_a$ whose structure morphisms are of finite type and quasi-separated ([Hu$_1$, 3.9]).

D) Cohomological dimension

A) Constructible sheaves

Let A be a noetherian ring. Our definition of a constructible A-module on the étale site of a rigid analytic variety is completely analogous to the definition of a constructible A-module on the étale site of a scheme. Recall that a sheaf F of A-modules on the etale site $X_{\acute{e}t}$ of a scheme X is called constructible if, for every affine open subscheme U of X, there is a finite covering $U = \bigcup_{i=1}^{n} U_i$ of U by locally closed constructible subschemes U_i of U such that, for $i = 1, \ldots, n$, the restriction $F|U_i$ of F to $(U_i)_{\acute{e}t}$ is a locally constant A-module of finite type ([SGA 4, IX.2.3]). For an affine scheme U, the subtoposes S of $U^{\sim}_{\acute{e}t}$ of the form $S = T^{\sim}_{\acute{e}t}$ with T a locally closed constructible subscheme of U are just the locally closed constructible subtoposes of $U^{\sim}_{\acute{e}t}$. In (0.2) we have seen that, for an affinoid rigid analytic variety U, the locally closed constructible subtoposes of $U^{\sim}_{\acute{e}t}$ correspond to the locally closed constructible subsets of U.

Definition 0.5.1. Let X be a rigid analytic variety. A sheaf F of A-modules on $X_{\acute{e}t}$ is called *constructible* if, for every affinoid open subspace U of X, there are locally closed constructible subsets $U_1, \ldots, U_n$ of U such that $U = \bigcup_{i=1}^{n} U_i$ and, for $i = 1, \ldots, n$, the restriction $F|U_i$ of F to the site $(U_i)_{\acute{e}t/U}$ is locally constant of finite type.

Let X be a quasi-compact quasi-separated rigid analytic variety. Then

- For every quasi-compact quasi-separated étale morphism $f: Y \to X$ and every constructible A-module F on $Y_{\acute{e}t}$, $f_!F$ is a constructible A-module on $X_{\acute{e}t}$.
- An A-module F on $X_{\acute{e}t}$ is constructible if and only if F is isomorphic to the cokernel of a morphism $(f_1)_!A_{Y_1} \to (f_2)_!A_{Y_2}$ where, for $i = 1, 2$, $f_i: Y_i \to X$ is a quasi-compact quasi-separated étale morphism of rigid analytic varieties and A_{Y_i} denotes the constant sheaf to A on $(Y_i)_{\acute{e}t}$.
- The category of constructible A-modules on $X_{\acute{e}t}$ is the smallest strictly full subcategory of the category of A-modules on $X_{\acute{e}t}$ which is closed under extensions and which contains, for every locally closed constructible subset U of X and every locally constant A-module F of finite type on $U_{\acute{e}t/X}$, the A-module $j_!F$ (where $j_!F$ is the A-module on $X_{\acute{e}t}$ with $(j_!F)|U_{\acute{e}t/X} = F$ and $(j_!F)|S_{\acute{e}t/X} = 0$ for every locally closed constructible subset S of X with $S \cap U = \emptyset$.)
- The functor $(F_i)_{i \in I} \mapsto \varinjlim_{i \in I} F_i$ is an equivalence from the category of ind-objects of constructible A-modules on $X_{\acute{e}t}$ to the category of A-modules on $X_{\acute{e}t}$.

- A constructible A-module F on $X_{\acute{e}t}$ is overconvergent ([JP]) if and only if F is locally constant.

(0.5.2) In this book, a standard argument to prove general statements on the cohomology of torsion sheaves on the étale site of a quasi-compact quasi-separated rigid analytic variety X will be the following. Since every torsion sheaf on $X_{\acute{e}t}$ is the inductive limit of a filtered inductive system of constructible sheaves, we can restrict ourselves to constructible sheaves. Then (since, for every constructible sheaf F on $X_{\acute{e}t}$, there are locally closed constructible subsets $X_1, \ldots, X_n$ of X such that $X = \bigcup_{i=1}^{n} X_i$ and $F|(X_i)_{\acute{e}t,X}$ is locally constant) one can very often reduce to constant $\mathbb{Z}/n\mathbb{Z}$-modules on sites of the form $S_{\acute{e}t,Y}$ where S is a special subset of an affinoid rigid analytic variety Y (cf. (0.2.2.ii, iii)). Then to such sheaves one can apply Theorem (0.2.4). Thus one can reduce the proof to a problem on the étale cohomology of affine schemes. (For example, we will use this reasoning in the proof of the acyclicity statement (0.1.1.ii) and in the proofs of the base change theorems.)

B) Finiteness of $H^n(X, F)$

We call a rigid analytic variety X over a complete non-archimedean field k *locally algebraic* if there is an admissible open covering $(X_i)_{i\in I}$ of X such that every X_i is isomorphic to an open subvariety of a rigid analytic variety which is associated to an algebraic variety over k. For example, every smooth rigid analytic variety is locally algebraic.

Proposition 0.5.3. *Let X be a quasi-compact quasi-separated locally algebraic rigid analytic variety over an algebraically closed complete non-archimedean field k. Let A be a noetherian ring which is annihilated by a natural number which is prime to $\mathrm{char}(k^\circ/k^{\circ\circ})$. Then, for every constructible A-module F on $X_{\acute{e}t}$ and every $n \in \mathbb{N}_0$, $H^n(X, F)$ is a finitely generated A-module.*

C) Finiteness of $R^n f_! F$

Proposition 0.5.4. *Let X, k, A be as in Proposition (0.5.3). Assume that X is separated. Then, for every constructible A-module F on $X_{\acute{e}t}$ and every $n \in \mathbb{N}_0$, $H^n_c(X, F)$ is a finitely generated A-module.*

Theorem 0.5.5. *Let $f: X \to Y$ be a separated quasi-compact smooth morphism of rigid analytic varieties over a complete non-archimedean field k, and let A be a noetherian ring which is annihilated by a natural number which is prime to $\mathrm{char}(k^\circ/k^{\circ\circ})$. Then, for every $n \in \mathbb{N}_0$, $R^n f_! F$ is a constructible A-module on $Y_{\acute{e}t}$.*

As a consequence of this theorem we obtain

Corollary 0.5.6. *Let $f: X \to Y$ be a proper smooth morphism of rigid analytic varieties over a complete non-archimedean field k, and let A be a noetherian ring which is annihilated by a natural number which is prime to $\mathrm{char}(k^\circ/k^{\circ\circ})$. Then, for every locally constant A-module F of finite type on $X_{\acute{e}t}$ and every $n \in \mathbb{N}_0$, $R^n f_* F$ is a locally constant A-module of finite type on $Y_{\acute{e}t}$.*

(1.1.13) A topological space X is called *spectral* if X is quasi-compact and has a basis of quasi-compact open subsets stable under finite intersections and if every irreducible closed subset is the closure of a unique point ([Ho]). A topological space X is called *locally spectral* if every point of X has an open neighbourhood which is spectral. A locally spectral space X is spectral if and only if X is quasi-separated (i.e., the intersection of two quasi-compact open subsets of X is quasi-compact) and quasi-compact. A subset T of a locally spectral space X is called

- *retro-compact* if, for every quasi-compact open subset U of X, the intersection $T \cap U$ is quasi-compact ([EGA*, 0.2.3.1])
- *globally constructible* if T is an element of the boolean algebra of subsets of X which is generated by the retro-compact open subsets of X ([EGA*, 0.2.3.2])
- *constructible* if every point of X has an open neighbourhood U such that $T \cap U$ is globally constructible in U ([EGA*, 0.2.3.10])
- *pro-constructible* (resp. *ind-constructible*) if every point of X has an open neighbourhood U such that $T \cap U$ is the intersection (resp. union) of constructible subsets of U ([EGA*, I.7.2.2])
- *locally constructible* (resp. *locally pro-constructible*, resp. *locally ind-constructible*) if T is a constructible (resp. pro-constructible, resp. ind-constructible) subset of an open subset of X
- *convex* if, for every $x, y, z \in X$, the relations $x \succ y \succ z$ and $x, z \in T$ imply $y \in T$.

Let X be a quasi-compact locally spectral space, $(T_i)_{i \in I}$ a family of pro constructible subsets of X, and S an ind-constructible subset of X such that $\bigcap_{i \in I} T_i \subseteq S$. Then there exists a finite subset J of I with $\bigcap_{i \in J} T_i \subseteq S$ ([Ho]).

Let $f : X \to Y$ be a continuous mapping between locally spectral spaces. f is called

- *quasi-separated* if, for every quasi-separated open subset V of Y, $f^{-1}(V)$ is quasi-separated
- *quasi-compact* if, for every quasi-compact open subset V of Y, $f^{-1}(V)$ is quasi-compact
- *spectral* if, for every quasi-compact quasi-separated open subsets $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, the restriction $f : U \to V$ is quasi-compact.

If f is quasi-compact and spectral then, for every pro-constructible subset T of X, $f(T)$ is pro-constructible in Y.

(1.1.14) The following simple properties of valuations are used freely

a) Let $v: K \longrightarrow \Gamma \cup \{0\}$ be a valuation of a field K and let $A \subseteq K$ be the valuation ring of v, i.e. $A = \{k \in K \mid v(k) \leq 1\}$. Then A is integrally closed in K. For every convex subgroup H of Γ, the set $\mathfrak{p}_H := \{k \in K \mid v(k) < H\}$ is a prime ideal of A. By $H \mapsto \mathfrak{p}_H$ we get a bijection from the set of all convex subgroups of Γ onto the set of all prime ideals of A. The set of prime ideals of A is totally ordered.

b) Let A be a valuation ring of a field K. For every prime ideal $\mathfrak{p}$ of A, the localization $A_\mathfrak{p}$ is a valuation ring of K and $\mathfrak{p}$ is the maximal ideal of $A_\mathfrak{p}$. By $\mathfrak{p} \mapsto A_\mathfrak{p}$ we get a bijection from the set of all prime ideals of A onto the set of all valuation rings of K which contain A. If $v: K \longrightarrow \Gamma \cup \{0\}$ is a valuation which corresponds to A and if $H \subseteq \Gamma$ is the convex subgroup which corresponds to $\mathfrak{p}$ then the mapping $K \to \Gamma/H \cup \{0\}$, $k \mapsto v(k) \mathrm{mod} H$ is a valuation of K which corresponds to the valuation ring $A_\mathfrak{p}$.

c) Let A be a valuation ring of a field K, and let $\lambda: A \to k$ be the natural mapping from A onto its residue class field. For every valuation ring B of k, $\lambda^{-1}(B)$ is a valuation ring of K. By $B \mapsto \lambda^{-1}(B)$ we get a bijection form the set of all valuation rings of k onto the set of all valuation rings of K which are contained in A.

d) Let K be a field and $A \subseteq K$ a local ring. Then there exists a valuation ring B of K which dominates A.

e) Let $K_1 \hookrightarrow K_2$ be an extension of fields. Let A_2 be a valuation ring of K_2 and put $A_1 := A_2 \cap K_1$. For $i = 1, 2$, let $\mathcal{G}_i$ be the set of all valuation rings of K_i which contain A_i and let $\mathcal{S}_i$ be the set of all valuation rings of K_i which are contained in A_i. Then the mappings $\varphi: \mathcal{G}_2 \to \mathcal{G}_1$, $B \mapsto B \cap K_1$ and $\psi: \mathcal{S}_2 \to \mathcal{S}_1$, $B \mapsto B \cap K_1$ are surjective (obvious by (a) - (d)). If K_2 is algebraic over K_1 then φ is bijective, the quotient Γ_2/Γ_1 of the value groups of A_2 and A_1 is a torsion group and the extension $k_1 \hookrightarrow k_2$ of the residue class fields of A_1 and A_2 is algebraic.

f) Let $K \hookrightarrow K_1$ and $K \hookrightarrow K_2$ be extensions of fields, and let A_1, A_2 be valuation rings of K_1, K_2 with $A_1 \cap K = A_2 \cap K$. Then there exist a composite K_3 of K_1 and K_2 over K and a valuation ring A_3 of K_3 with $A_1 = A_3 \cap K_1$ and $A_2 = A_3 \cap K_2$.

g) Let A be a valuation ring of a field K with corresponding valuation $v: K \to \Gamma \cup \{0\}$. The set of all sets of the form $\{x \in K \mid v(x) < \gamma\}$ $(\gamma \in \Gamma)$ is a fundamental system of neighbourhoods of 0 of a field topology of K (the valuation topology of A). The completion $\hat{K}$ of K is a field, the completion $\hat{A}$ of A is a valuation ring of $\hat{K}$, and the topology of $\hat{K}$ is the valuation topology of $\hat{A}$. The value groups and the residue class fields of A and $\hat{A}$ are equal. For every valuation ring B of K with $A \subseteq B \subsetneq K$, the valuation topologies of A and B on K coincide.

1.2 Morphisms of finite type

A ring homomorphism $f: A \to B$ between f-adic rings is called *adic* if there are rings of definition A_0, B_0 of A, B and an ideal of definition I of A_0 such that $f(A_0) \subseteq B_0$ and $f(I) \cdot B_0$ is an ideal of definition of B_0. Then, for all rings of definition A_0, B_0 of A, B with $f(A_0) \subseteq B_0$ and every ideal of definition I of A_0, $f(I) \cdot B_0$ is an ideal

of definition of B_0 [Hu$_1$, 1.8.ii]).

A morphism of adic spaces $f: X \to Y$ is called *adic* if, for every $x \in X$, there exist open affinoid subspaces U, V of X, Y such that $x \in U$, $f(U) \subseteq V$ and $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is adic. (Then, for every pair U, V of open affinoid subspaces of X, Y with $f(U) \subseteq V$, $\mathcal{O}_Y(V) \to \mathcal{O}_X(V)$ is adic [Hu, 3.2]). f is adic if and only if $f(X_a) \subseteq Y_a$ [loc. cit.]. In particular, if Y is analytic then every morphism $f: X \to Y$ is adic.

In [Hu, §3] we defined ring homomorphisms of *topologically finite type* between f-adic rings (resp. affinoid rings).

[Huber 94] Definition, p. 535

Definition 1.2.1. Let $f: X \to Y$ be a morphism of adic spaces.

i) f is called *locally of weakly finite type* if, for every $x \in X$, there exist open affinoid subspaces U, V of X, Y such that $x \in U$, $f(U) \subseteq V$ and the ring homomorphism of f-adic rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of topologically finite type.

ii) f is called *locally of $^+$weakly finite type* if, for every $x \in X$, there exist open affinoid subspaces U, V of X, Y and a finite subset E of $\mathcal{O}_X(U)$ such that $x \in U$, $f(U) \subseteq V$, the ring homomorphism of f-adic rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of topologically finite type and $\mathcal{O}_X^+(U)$ is the smallest ring of integral elements of $\mathcal{O}_X(U)$ which contains E and the image of $\mathcal{O}_Y^+(V)$ in $\mathcal{O}_X(U)$ (i.e., $\mathcal{O}_X^+(U)$ is the integral closure of $\mathcal{O}_Y^+(V)[E \cup \mathcal{O}_X(U)^{\circ\circ}]$ in $\mathcal{O}_X(U)$).

iii) f is called *locally of finite type* if, for every $x \in X$, there exist open affinoid subspaces U, V of X, Y such that $x \in U$, $f(U) \subseteq V$ and the ring homomorphism of affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type.

iv) f is called *of finite type* (resp. *of $^+$weakly finite type*, resp. *of weakly finite type*) if f is quasi-compact and locally of finite type (resp. locally of $^+$weakly finite type, resp. locally of weakly finite type).

v) f is called *locally of finite presentation* if, for every $x \in X$, there exist open affinoid subspaces U, V of X, Y such that $x \in U$, $f(U) \subseteq V$, the morphism of affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type and, in case that the topology of $\mathcal{O}_Y(V)$ is discrete, the ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite presentation.

Then {morphisms locally of finite presentation} $\subseteq$ {morphisms locally of finite type} $\subset$ {morphisms locally of $^+$weakly finite type} $\subseteq$ {morphisms locally of weakly finite type} $\subset$ {adic morphisms}

If $f: X \to Y$ is locally of finite presentation (resp. locally of finite type, resp. locally of $^+$weakly finite type, resp. locally of weakly finite type) then, for every pair U, V of open subspaces of X, Y with $f(U) \subseteq V$, the restriction $f: U \to V$ is locally of finite presentation (resp. locally of finite type, resp. locally of $^+$weakly finite type, resp. locally of weakly finite type). If $f: X \to Y$ and $g: Y \to Z$ are locally of finite presentation (resp. locally of finite type, resp. locally of $^+$weakly finite type, resp. locally of weakly finite type) then also $g \circ f$ is locally of finite presentation (resp. locally of finite type, resp. locally of $^+$weakly finite type, resp. locally of weakly finite type).

If $X \to Y$ is of finite type and X, Y are affinoid then the ring homomorphism of affinoid rings $(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y)) \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ is of topologically finite type ([Hu$_4$, 3.3.23 and 3.8.15], [Hu$_2$]).

If $f: X \to Y$ is a morphism of rigid analytic varieties then the associated morphism of adic spaces $r(f): r(X) \to r(Y)$ is locally of finite type ([Hu, 4.3]).

[Hub94] Prop. 2.1 p. 522 [Hub 94] Lemma 3.3, p. 533

Proposition 1.2.2. *Let $f: X \to Z$ and $g: Y \to Z$ be morphisms of adic spaces. The fibre product of f and g in the category of adic spaces exists if one of the following conditions is satisfied*

a) *f is locally of finite type*

b) *f is locally of weakly finite type and g is adic.*

Proof. 1) Assume that b) is satisfied. We may assume that X, Y, Z are affinoid. Put $A := (\mathcal{O}_Z(Z), \mathcal{O}_Z^+(Z))$, $B := (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ and $C := (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$, and let $\varphi: A \to B$ and $\psi: A \to C$ be the homomorphisms induced by g and f. Then $A^\flat \to B^\flat$ is adic, and we may assume that $A^\flat \to C^\flat$ is of topologically finite type. Let A_0, B_0, C_0 be rings of definition of $A^\flat, B^\flat, C^\flat$ with $\varphi(A_0) \subseteq B_0$ and $\psi(A_0) \subseteq C_0$, and let I be an ideal of definition of A_0. Put $D^\flat := C^\flat \otimes_{A^\flat} B^\flat$. Let D_0 be the image of $C_0 \otimes_{A_0} B_0$ in $D^\flat$, and let D^+ be the integral closure of $C^+ \otimes_{A^+} B^+$ in $D^\flat$. We equip $D^\flat$ with the group topology such that $\{I^n \cdot D_0 \mid n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of zero. Then $D := (D^\flat, D^+)$ is an affinoid ring ($(D^\flat)^\wedge$ is of topologically finite type over $B^\flat$), and the natural diagram

$$\begin{array}{ccc} D & \longleftarrow & C \\ \uparrow & & \uparrow \psi \\ B & \xleftarrow{\varphi} & A \end{array}$$

is cocartesian in the category of affinoid rings. Hence the fibre product of f and g exists.

2) Assume that a) is satisfied. We may assume that X, Y, Z are affinoid. Let A, B, C be complete affinoid rings with $Z = \mathrm{Spa}A$, $Y = \mathrm{Spa}B$, $X = \mathrm{Spa}C$, and let $\varphi: A \to B$ and $\psi: A \to C$ be the ring homomorphisms corresponding to g and f. Furthermore we may assume that ψ is of topologically finite type. So we choose a representation $C = A\langle T\rangle_M / I = A\langle T_1, \ldots, T_n\rangle_{M_1, \ldots, M_n} / I$ where $M_1, \ldots, M_n$ are finite subsets of $A^\flat$ such that $M_i \cdot A^\flat$ is open in $A^\flat$ for $i = 1, \ldots, n$ and I is an ideal of $A\langle T\rangle_M$ ([Hu, §3]). Furthermore we choose a finite subset L of $(B^\flat)^{\circ\circ}$ such that $(\varphi(M_i) \cup L^m) \cdot B^\flat$ is open in $B^\flat$ for every $m \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$ (for example, let L be a finite set of generators of an ideal of definition of a ring of definition of $B^\flat$). Then, for every $m \in \mathbb{N}$, we have the affinoid ring $B\langle T\rangle_{\varphi(M)\cup L^m} := B\langle T_1, \ldots, T_n\rangle_{\varphi(M_1)\cup L^m, \ldots, \varphi(M_n)\cup L^m}$. Let $\psi': A \to A\langle T\rangle_M$ and $\varphi'_m: B \to B\langle T\rangle_{\varphi(M)\cup L^m}$ be the natural morphisms. By [Hu, 3.5.a] there exists a unique continuous morphism $\varphi'_m: A\langle T\rangle_M \to B\langle T\rangle_{\varphi(M)\cup L^m}$ such that $\varphi'_m(T_i) = T_i$ $(i = 1, \ldots, n)$ and the diagram

of definition of $\mathcal{B}_0$ [Hu$_1$, 1.8.ii]).

A morphism of adic spaces $f: X \to Y$ is called *adic* if, for every $x \in X$, there exist open affinoid subspaces U, V of X, Y such that $x \in U$, $f(U) \subseteq V$ and $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is adic. (Then, for every pair U, V of open affinoid subspaces of X, Y with $f(U) \subseteq V$, $\mathcal{O}_Y(V) \to \mathcal{O}_X(V)$ is adic [Hu, 3.2]). f is adic if and only if $f(X_a) \subseteq Y_a$ [loc. cit.]. In particular, if Y is analytic then every morphism $f: X \to Y$ is adic.

In [Hu, §3] we defined ring homomorphisms of *topologically finite type* between f-adic rings (resp. affinoid rings).

Definition 1.2.1. Let $f: X \to Y$ be a morphism of adic spaces.

[Huber 94] Definition, p. 535

i) f is called *locally of weakly finite type* if, for every $x \in X$, there exist open affinoid subspaces U, V of X, Y such that $x \in U$, $f(U) \subseteq V$ and the ring homomorphism of f-adic rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of topologically finite type.

ii) f is called *locally of $^+$weakly finite type* if, for every $x \in X$, there exist open affinoid subspaces U, V of X, Y and a finite subset E of $\mathcal{O}_X(U)$ such that $x \in U$, $f(U) \subseteq V$, the ring homomorphism of f-adic rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of topologically finite type and $\mathcal{O}_X^+(U)$ is the smallest ring of integral elements of $\mathcal{O}_X(U)$ which contains E and the image of $\mathcal{O}_Y^+(V)$ in $\mathcal{O}_X(U)$ (i.e., $\mathcal{O}_X^+(U)$ is the integral closure of $\mathcal{O}_Y^+(V)[E \cup \mathcal{O}_X(U)^{\circ\circ}]$ in $\mathcal{O}_X(U)$).

iii) f is called *locally of finite type* if, for every $x \in X$, there exist open affinoid subspaces U, V of X, Y such that $x \in U$, $f(U) \subseteq V$ and the ring homomorphism of affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type.

iv) f is called *of finite type* (resp. *of $^+$weakly finite type*, resp. *of weakly finite type*) if f is quasi-compact and locally of finite type (resp. locally of $^+$weakly finite type, resp. locally of weakly finite type).

v) f is called *locally of finite presentation* if, for every $x \in X$, there exist open affinoid subspaces U, V of X, Y such that $x \in U$, $f(U) \subseteq V$, the morphism of affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type and, in case that the topology of $\mathcal{O}_Y(V)$ is discrete, the ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite presentation.

Then {morphisms locally of finite presentation} $\subseteq$ {morphisms locally of finite type} $\subseteq$ {morphisms locally of $^+$weakly finite type} $\subseteq$ {morphisms locally of weakly finite type} $\subseteq$ {adic morphisms}.

If $f: X \to Y$ is locally of finite presentation (resp. locally of finite type, resp. locally of $^+$weakly finite type, resp. locally of weakly finite type) then, for every pair U, V of open subspaces of X, Y with $f(U) \subseteq V$, the restriction $f: U \to V$ is locally of finite presentation (resp. locally of finite type, resp. locally of $^+$weakly finite type, resp. locally of weakly finite type). If $f: X \to Y$ and $g: Y \to Z$ are locally of finite presentation (resp. locally of finite type, resp. locally of $^+$weakly finite type, resp. locally of weakly finite type) then also $g \circ f$ is locally of finite presentation (resp. locally of finite type, resp. locally of $^+$weakly finite type, resp. locally of weakly finite type).

$$\begin{array}{ccc} B\langle T\rangle_{\varphi(M)\cup L^m} & \xleftarrow{\varphi'_m} & A\langle T\rangle_M \\ \uparrow{\scriptstyle \psi'_m} & & \uparrow{\scriptstyle \psi'} \\ B & \xleftarrow[\varphi]{} & A \end{array}$$

commutes. Likewise by [loc. cit.] there exists, for $m_1 \geq m_2$, a unique continuous morphism $\lambda'_{m_1,m_2}: B\langle T\rangle_{\varphi(M)\cup L^{m_1}} \to B\langle T\rangle_{\varphi(M)\cup L^{m_2}}$ such that $\lambda'_{m_1,m_2}(T_i) = T_i \quad (i = 1,\ldots,n)$ and $\psi'_{m_2} = \lambda'_{m_1,m_2} \circ \psi'_{m_1}$. Obviously, $\varphi'_{m_2} = \lambda'_{m_1,m_2} \circ \varphi'_{m_1}$. Let J_m be the closure of the ideal of $(B\langle T\rangle_{\varphi(M)\cup L^m})^\flat$ generated by $\varphi'_m(I)$. Then φ'_m induces a continuous morphism $\varphi_m: A\langle T\rangle_M/I \to B\langle T\rangle_{\varphi(M)\cup L^m}/J_m$, ψ'_m induces a continuous morphism $\psi_m: B \to B\langle T\rangle_{\varphi(M)\cup L^m}/J_m$, and λ'_{m_1,m_2} induces a continuous morphism $\lambda_{m_1,m_2}: B\langle T\rangle_{\varphi(M)\cup L^{m_1}}/J_{m_1} \to B\langle T\rangle_{\varphi(M)\cup L^{m_2}}/J_{m_2}$. The diagram

$$(\star) \qquad \begin{array}{ccc} B\langle T\rangle_{\varphi(M)\cup L^m}/J_m & \xleftarrow{\varphi_m} & A\langle T\rangle_M/I = C \\ \uparrow{\scriptstyle \psi_m} & & \uparrow{\scriptstyle \psi} \\ B & \xleftarrow[\varphi]{} & A \end{array}$$

commutes, and $\psi_{m_2} = \lambda_{m_1,m_2} \circ \psi_{m_1}$ and $\varphi_{m_2} = \lambda_{m_1,m_2} \circ \varphi_{m_1}$. Put $W_m := \operatorname{Spa} B\langle T\rangle_{\varphi(M)\cup L^m}/J_m$. The diagram $(\star)$ from above induces a commutative diagram of adic spaces

$$\begin{array}{ccc} W_m & \xrightarrow{g_m} & X \\ \downarrow{\scriptstyle f_m} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

and λ_{m_1,m_2} induces a morphism $t_{m_1,m_2}: W_{m_2} \to W_{m_1}$. We have $f_{m_2} = f_{m_1} \circ t_{m_1,m_2}$ and $g_{m_2} = g_{m_1} \circ t_{m_1,m_2}$. It is easily seen that $t_{m_1,m_2}: W_{m_2} \to W_{m_1}$ is an open embedding (more precisely, t_{m_1,m_2} induces an isomorphism from W_{m_2} onto the rational subspace $\{x \in W_{m_1} \mid v_x(fT_i) \leq 1 \text{ for } i = 1,\ldots,n \text{ and } f \in L^{m_2}\}$ of W_{m_1}). So there exists in the category of adic spaces the inductive limit W of the inductive system $(W_m, m \in \mathbb{N}; t_{m_1,m_2}, m_1 \geq m_2)$. The $f_m, m \in \mathbb{N}$ induce a morphism $f': W \to Y$, and the $g_m, m \in \mathbb{N}$ induce a morphism $g': W \to X$. The diagram

$$(\star\star) \qquad \begin{array}{ccc} W & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

commutes. We state that this diagram is cartesian in the category of adic spaces. In order to check this let $V = \mathrm{Spa}D$ be an affinoid adic space and $p: V \to X$ and $q: V \to Y$ morphisms with $f \circ p = g \circ q$. The morphisms p and q correspond to continuous morphisms $\sigma: C \to D$ and $\tau: B \to D$ with $\sigma \circ \psi = \tau \circ \varphi$. For $i = 1, \ldots, n$, let $\overline{T}_i \in C^\flat$ be the image of $T_i \in (A\langle T\rangle_M)^\flat$ in $(A\langle T\rangle_M)^\flat/I$. For every $a \in M_i$, we have $\tau(\varphi(a)) \cdot \sigma(\overline{T}_i) = \sigma(\psi(a)) \cdot \sigma(\overline{T}_i) = \sigma(\psi(a)\overline{T}_i) \in D^+$. Since L is a finite subset of $(B^\flat)^{\circ\circ}$, there exists a $m \in \mathbb{N}$ such that, for every $\ell \in L^m$ and $i \in \{1, \ldots, n\}$, $\tau(\ell) \cdot \sigma(\overline{T}_i) \in D^+$. Then by [Hu, 3.5] there exists a unique continuous morphism $\varepsilon': B\langle T\rangle_{\varphi(M)\cup L^m} \to D$ with $\tau = \varepsilon' \circ \psi'_m$ and $\varepsilon'(T_i) = \sigma(\overline{T}_i)$ for $i = 1, \ldots, n$. The uniqueness statement of [loc. cit.] implies that $\varepsilon' \circ \varphi'_m = \sigma \circ \delta$, where $\delta: A\langle T\rangle_M \to A\langle T\rangle_M/I = C$ is the natural morphism. Hence $\varepsilon'(J_m) = 0$, and so ε' factorizes through a continuous morphism $\varepsilon: B\langle T\rangle_{\varphi(M)\cup L^m}/J_m \to D$. This shows that the diagram (**) is cartesian. □

Corollary 1.2.3. *Let*

$$\begin{array}{ccc} S & \xrightarrow{v} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[g]{} & Y \end{array}$$

be a cartesian diagram of adic spaces.

i) *If f is locally of finite presentation (resp. locally of finite type) then u is locally of finite presentation (resp. locally of finite type).*

ii) *If f is locally of $^+$weakly finite type and g is adic then u is locally of $^+$weakly finite type.*

iii) *Assume that f is locally of weakly finite type and g is adic. Then*

a) *u is locally of weakly finite type and v is adic.*

b) *If f is quasi-compact (resp. quasi-separated) then u is quasi-compact (resp. quasi-separated).*

c) *If g is quasi-compact (resp. quasi-separated) then v is quasi-compact (resp. quasi-separated).*

d) *For every $x \in X$ and $t \in T$ with $f(x) = g(t)$, there exists a $s \in S$ with $x = v(s)$ and $t = u(s)$.*

Proof. All assertions follow from the proof of (1.2.2). For the proof of (iii.d), cf. proof of [Hu, 3.9.i] □

(1.2.4) Let $f: X \to Y$ be a morphism of adic spaces which is locally of weakly finite type, and let y be a point of Y. The natural morphism $\mathrm{Spa}(\kappa(y)) \to Y$ ((1.1.8)) is adic. Hence by (1.2.2) there exists the fibre product

$$\begin{array}{ccc} B\langle T\rangle_{\varphi(M)\cup L^m} & \xleftarrow{\varphi'_m} & A\langle T\rangle_M \\ \uparrow{\scriptstyle \psi'_m} & & \uparrow{\scriptstyle \psi'} \\ B & \xleftarrow[\varphi]{} & A \end{array}$$

commutes. Likewise by [loc. cit.] there exists, for $m_1 \geq m_2$, a unique continuous morphism $\lambda'_{m_1,m_2}: B\langle T\rangle_{\varphi(M)\cup L^{m_1}} \to B\langle T\rangle_{\varphi(M)\cup L^{m_2}}$ such that $\lambda'_{m_1,m_2}(T_i) = T_i \quad (i = 1, \ldots, n)$ and $\psi'_{m_2} = \lambda'_{m_1,m_2} \circ \psi'_{m_1}$. Obviously, $\varphi'_{m_2} = \lambda'_{m_1,m_2} \circ \varphi'_{m_1}$. Let J_m be the closure of the ideal of $(B\langle T\rangle_{\varphi(M)\cup L^m})^\flat$ generated by $\varphi'_m(I)$. Then φ'_m induces a continuous morphism $\varphi_m: A\langle T\rangle_M/I \to B\langle T\rangle_{\varphi(M)\cup L^m}/J_m$, ψ'_m induces a continuous morphism $\psi_m: B \to B\langle T\rangle_{\varphi(M)\cup L^m}/J_m$, and λ'_{m_1,m_2} induces a continuous morphism $\lambda_{m_1,m_2}: B\langle T\rangle_{\varphi(M)\cup L^{m_1}}/J_{m_1} \to B\langle T\rangle_{\varphi(M)\cup L^{m_2}}/J_{m_2}$. The diagram

$$(\star) \qquad \begin{array}{ccc} B\langle T\rangle_{\varphi(M)\cup L^m}/J_m & \xleftarrow{\varphi_m} & A\langle T\rangle_M/I = C \\ \uparrow{\scriptstyle \psi_m} & & \uparrow{\scriptstyle \psi} \\ B & \xleftarrow[\varphi]{} & A \end{array}$$

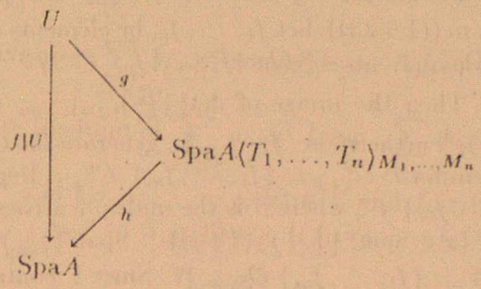
commutes, and $\psi_{m_2} = \lambda_{m_1,m_2} \circ \psi_{m_1}$ and $\varphi_{m_2} = \lambda_{m_1,m_2} \circ \varphi_{m_1}$. Put $W_m := \mathrm{Spa}B\langle T\rangle_{\varphi(M)\cup L^m}/J_m$. The diagram $(\star)$ from above induces a commutative diagram of adic spaces

$$\begin{array}{ccc} W_m & \xrightarrow{g_m} & X \\ \downarrow{\scriptstyle f_m} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

and λ_{m_1,m_2} induces a morphism $t_{m_1,m_2}: W_{m_2} \to W_{m_1}$. We have $f_{m_2} = f_{m_1} \circ t_{m_1,m_2}$ and $g_{m_2} = g_{m_1} \circ t_{m_1,m_2}$. It is easily seen that $t_{m_1,m_2}: W_{m_2} \to W_{m_1}$ is an open embedding (more precisely, t_{m_1,m_2} induces an isomorphism from W_{m_2} onto the rational subspace $\{x \in W_{m_1} \mid v_x(T_i) \leq 1$ for $i = 1, \ldots, n$ and $t \in L^{m_2}\}$ of W_{m_1}). So there exists in the category of adic spaces the inductive limit W of the inductive system $(W_m, m \in \mathbb{N}; t_{m_1,m_2}, m_1 \geq m_2)$. The $f_m, m \in \mathbb{N}$ induce a morphism $f': W \to Y$, and the $g_m, m \in \mathbb{N}$ induce a morphism $g': W \to X$. The diagram

$$(\star\star) \qquad \begin{array}{ccc} W & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

Corollary 1.6.10. *Let A be an affinoid ring and $f: X \to \mathrm{Spa}A$ a morphism of adic spaces. f is smooth if and only if, for every $x \in X$, there exists a commutative diagram*



$(\star)$

where U is an open neighbourhood of x in X, $M_1, \ldots, M_n$ are finite subsets of A° with $M_i \cdot A^\circ$ open in A° for $i = 1, \ldots, n$, h is the natural morphism and g is étale. If A is Tate and f is smooth then, for every $x \in X$, there exists a commutative diagram $(\star)$ with $M_1 = \ldots = M_n = \{1\}$.

Proof. Put $Y := \mathrm{Spa}A$. Assume that f is smooth. By (1.6.9.i), for every $x \in X$, there exist an open affinoid neighbourhood U of x in X and $s_1, \ldots, s_n \in \mathcal{O}_X(U)$ such that $\Omega_{X|Y}|U$ is a free $\mathcal{O}_X|U$-module with basis $d_{X|Y}(s_1), \ldots, d_{X|Y}(s_n) \in \Omega_{X|Y}(U)$. Choose finite subsets $M_1, \ldots, M_n$ of A° such that $M_i \cdot A^\circ$ is open in A° and $M_i \cdot s_i \in \mathcal{O}_X^+(U)$ for $i = 1, \ldots, n$. Then there exists a commutative diagram $(\star)$ with $g^*(T_i) = s_i$ for $i = 1, \ldots, n$ ([Hu, 3.5.i]). By (1.6.9.iii) and (1.6.2.i) g is étale. Assume that A° is Tate. Let s be a topologically nilpotent unit of A°. Choose a $\ell \in \mathbb{N}$ such that $s^\ell \cdot s_i \in \mathcal{O}_X^+(U)$ for $i = 1, \ldots, n$. Replacing s_i by $s^\ell \cdot s_i$, we may assume that $s_i \in \mathcal{O}_X^+(U)$ for $i = 1, \ldots, n$. Then we can choose $M_1 = \ldots = M_n = \{1\}$. □

1.7 Étale morphisms

Proposition 1.7.1. *Let A be an affinoid ring (which is not necessarily complete), and let $s: X \to Y := \mathrm{Spa}A$ be a morphism between affinoid adic spaces. Then the following conditions are equivalent*

i) *s is étale*

ii) *There exist finite subsets $T_1, \ldots, T_n$ of A° with $T_i \cdot A^\circ$ open in A° for $i = 1, \ldots, n$ and there exist $f_1, \ldots, f_n \in A^\circ\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}$ such that the following holds: If we put $B := A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}$ and $I := f_1 B^\circ + \ldots + f_n B^\circ$ then the image of the determinant $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n} \in B^\circ$ in B°/I is a unit of B°/I and X is Y-isomorphic to $\mathrm{Spa}B/I$.*

iii) *There exist finite subsets $T_1, \ldots, T_n$ of A° with $T_i \cdot A^\circ$ open in A° for $i = 1, \ldots, n$ and there exist $f_1, \ldots, f_n \in A^\circ[X_1, \ldots, X_n]$ such that the following holds: If we put $B := A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}$ and $I := f_1 B^\circ + \ldots + f_n B^\circ$ then the image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n} \in A^\circ[X_1, \ldots, X_n]$ in B°/I is a unit of B°/I and X is Y-isomorphic to $\mathrm{Spa}B/I$.*

Proof. (i) $\Longrightarrow$ (ii): We choose finite subsets $T_1, \ldots, T_m$ of $A^\flat$ with $T_i \cdot A^\flat$ open for $i = 1, \ldots, m$ and an ideal J of $C := A\langle X_1, \ldots, X_m\rangle_{T_1, \ldots, T_m}$ such that X is Y-isomorphic to $\mathrm{Spa}C/J$. The natural mapping of $C^\flat/J$-modules $e: J/J^2 \to \Omega_{C^\flat|A^\flat} \otimes_{C^\flat} C^\flat/J$ is bijective (cf. (1.6.3.ii), (1.6.8), (1.6.9.ii)), and $\Omega_{C^\flat|A^\flat} \otimes_{C^\flat} C^\flat/J$ is a free $C^\flat/J$-module of rank m ((1.6.2.i)). Let $f_1, \ldots, f_m$ be elements of J such that $e(f_1), \ldots, e(f_m)$ is a basis of $\Omega_{C^\flat|A^\flat} \otimes_{C^\flat} C^\flat/J$. ($f_1, \ldots, f_m \in J/J^2$ denote the residue classes of $f_1, \ldots, f_m$). Then the image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,m} \in C^\flat$ in $C^\flat/J$ is a unit of $C^\flat/J$. ((1.6.2.i)). Furthermore, $f_1, \ldots, f_m$ generate the $C^\flat/J$-module J/J^2. Then by Nakayama's Lemma $J \cdot C^\flat_{1+J} = \{f_1, \ldots, f_m\} \cdot C^\flat_{1+J}$. Hence there exists a $j \in J$ with $J \cdot C^\flat_S = \{f_1, \ldots, f_m\} \cdot C^\flat_S$ where S is the multiplicative system $\{(1+j)^n \mid n \in \mathbb{N}_0\}$ of $C^\flat$. Then $V := \{x \in \mathrm{Spa}C \mid 1 \le v_x(1+j)\} = \mathrm{Spa}C\langle\frac{1}{1+j}\rangle$ is a rational subset of $\mathrm{Spa}C$ with $J \cdot \mathcal{O}_{\mathrm{Spa}C}|V = \{f_1, \ldots, f_m\} \cdot \mathcal{O}_{\mathrm{Spa}C}|V$. Since V contains the subspace $\mathrm{Spa}C/J$ of $\mathrm{Spa}C$, we obtain $X = \mathrm{Spa}C/J = \mathrm{Spa}C\langle\frac{1}{1+j}\rangle/J \cdot C\langle\frac{1}{1+j}\rangle = \mathrm{Spa}C\langle\frac{1}{1+j}\rangle/(f_1, \ldots, f_m)$. Let X_{m+1} be a variable, and put $f_{m+1} = 1 - (1+j)X_{m+1}$. Then $C\langle\frac{1}{1+j}\rangle = C\langle X_{m+1}\rangle/(f_{m+1})$, and hence $X = \mathrm{Spa}C\langle X_{m+1}\rangle/(f_1, \ldots, f_{m+1}) = \mathrm{Spa}B/I$ with $B := A\langle X_1, \ldots, X_{m+1}\rangle_{T_1, \ldots, T_m, \{1\}}$ and $I := f_1 B^\flat + \ldots + f_{m+1} B^\flat$. The image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,m+1} = -(1+j)\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,m} \in B^\flat$ in $B^\flat/I$ is a unit of $B^\flat/I$.
(ii) $\Longrightarrow$ (iii): We need some preparations which are given in the following points (1) to (4).

(1) Let A be a complete f-adic ring and $f_i = \sum_{\nu \in \mathbb{N}_0^n} a_{i,\nu} X^\nu$, $i = 1, \ldots, n$ elements of $A[[X_1, \ldots, X_n]]$ such that, for every $i = 1, \ldots, n$, there exists a bounded subset K of A with $a_{i,\nu} \in K^{|\nu|} = K \cdot \ldots \cdot K$ ($|\nu|$-fold product) for every $\nu \in \mathbb{N}_0^n$, $a_{i,(0,\ldots,0)} = 0$ and $\det(a_{i,\epsilon_j})_{i,j=1,\ldots,n} \in A^*$ (ϵ_j denotes the j-th unit vector $(0, \ldots, 1, \ldots, 0)$). Then there exists a neighbourhood U of 0 in $A^{\times n} = A \times \ldots \times A$ such that, for every $i = 1, \ldots, n$ and $x \in U$, the infinite series $\sum_{\nu \in \mathbb{N}_0^n} a_{i,\nu} x^\nu$ converges in A (the limit is denoted by $f_i(x)$) and $\{(f_1(x), \ldots, f_n(x)) \mid x \in U'\}$ is a neighbourhood of 0 in $A^{\times n}$ for any neighbourhood U' of 0 in U.

Proof. Let $(b_{ij})_{i,j=1,\ldots,n} \in \mathrm{Mat}(n \times n, A)$ be the inverse matrix of $(a_{i,\epsilon_j})_{i,j=1,\ldots,n}$. It is enough to prove the assertion for $f_1', \ldots, f_n' \in A[[X_1, \ldots, X_n]]$ with $f_i' := \sum_{j=1}^n b_{ij} f_j$. Hence we may assume $a_{i,\epsilon_i} = 1$ and $a_{i,\epsilon_j} = 0$ for $i \neq j$. By [B, III.4.4 Prop. 5] there exist $g_i = \sum_{\nu \in \mathbb{N}_0^n} b_{i,\nu} X^\nu \in A[[X_1, \ldots, X_n]], i = 1, \ldots, n$ with $g_i(0) = 0$ and $g_i(f_1, \ldots, f_n) = X_i = f_i(g_1, \ldots, g_n)$ for $i = 1, \ldots, n$. Let L be a bounded subset of A with $a_{i,\nu} \in L^{|\nu|}$ for $i = 1, \ldots, n$, $\nu \in \mathbb{N}_0^n$, and put $M := (L \cup \{1\})^2$. Then M is a bounded subset of A, and $a_{i,\nu} \in M^{|\nu|-1}$ for $i = 1, \ldots, n$, $\nu \in \mathbb{N}_0^n$ (put $M^{-1} = \{0\}$). By induction on $|\nu|$ one can prove $b_{i,\nu} \in M^{|\nu|}$ for $i = 1, \ldots, n$, $\nu \in \mathbb{N}_0^n$. Let P be a neighbourhood of 0 in A such that $(P^n \mid n \in \mathbb{N})$ is a zero sequence (i.e. to every neighbourhood W of 0 in A there exists a $k \in \mathbb{N}$ with $P^\ell \subset W$ for $\ell > k$).

let S be a neighbourhood of 0 in A with $S \cdot L \subseteq P$, $S + S \subseteq S$, $S \cdot S \subseteq S$, and let T be a neighbourhood of 0 in A with $T \cdot M \subseteq S \cap P$. For every $x \in S^{\times n}$ and $i = 1, \ldots, n$, the series $\sum_{\nu \in \mathbb{N}_0^n} a_{i,\nu} x^\nu$ converges in A (since $a_{i,\nu} \in L^{|\nu|}$), and, for every $x \in T^{\times n}$, the series $\sum_{\nu \in \mathbb{N}_0^n} b_{i,\nu} x^\nu$ converges in A (since $b_{i,\nu} \in M^{|\nu|}$). Let $f_i(x)$ (resp. $g_i(x)$) denote the limit of $\sum_{\nu \in \mathbb{N}_0^n} a_{i,\nu} x^\nu$ (resp. $\sum_{\nu \in \mathbb{N}_0^n} b_{i,\nu} x^\nu$). Then, for evey $x = (x_1, \ldots, x_n) \in T^{\times n}$ and $i = 1, \ldots, n$, $g_i(x) \in S$ and $f_i(g_1(x), \ldots, g_n(x)) = x_i$ (the latter since $f_i(g_1, \ldots, g_n) = X_i$). Hence $\{(f_1(x), \ldots, f_n(x)) \mid x \in S^{\times n}\}$ is a neighbourhood of 0 in $A^{\times n}$.

(2) Let A be a f-adic ring, $T_1, \ldots, T_n$ finite subsets of A with $T_i \cdot A$ open in A, B a complete f-adic A-algebra such that the structure morphism $f: A \to B$ is adic, $g: A\langle X \rangle_T := A\langle X_1, \ldots, X_n \rangle_{T_1, \ldots, T_n} \to B$ an adic A-algebra homomorphism, $f_1, \ldots, f_n$ elements of $A\langle X \rangle_T$ such that the image of the determinant $\det(\frac{\partial f_\nu}{\partial X_\mu})_{\mu,\nu=1,\ldots,n} \in A\langle X \rangle_T$ in B is a unit of B, and U a neighbourhood of 0 in B. Then there exists a neighbourhood V of 0 in B such that, for every $(v_1, \ldots, v_n) \in V^{\times n}$, there exists a continuous A-algebra homomorphism $h: A\langle X \rangle_T \to B$ with $h(f_i) = g(f_i) + v_i$ and $h(X_i) - g(X_i) \in U$ for $i = 1, \ldots, n$.

Proof. Since $f: A \to B$ is adic, $f(T_i) \cdot B$ is open in B for $i = 1, \ldots, n$, and since $g: A\langle X \rangle_T \to B$ is adic, $f(t) \cdot g(X_i)$ is power-bounded in B for every $i \in \{1, \ldots, n\}$, $t \in T_i$. Hence by the universal property ([Hu, 3.3.i]) there exist continuous ring homomorphisms $r: A\langle X \rangle_T \to B\langle X \rangle_{f(T)} := B\langle X_1, \ldots, X_n \rangle_{f(T_1), \ldots, f(T_n)}$ and $s: B\langle X \rangle_{f(T)} \to B\langle X \rangle_{f(T)}$ such that r extends f and $r(X_i) = X_i$ for $i = 1, \ldots, n$, and $s(b) = b$ for every $b \in B$ and $s(X_i) = g(X_i) + X_i$ for $i = 1, \ldots, n$. We put $m_i := s(r(f_i) - g(f_i)) \in B\langle X \rangle_{f(T)}$ for $i = 1, \ldots, n$. Then $\det(\frac{\partial m_\nu}{\partial X_\mu}(0))_{\mu,\nu=1,\ldots,n} \in B$ is a unit of B (since $B^* \ni g(\det \frac{\partial f_\nu}{\partial X_\mu}) = \det \frac{\partial r(f_\nu)}{\partial X_\mu}(g(X_1), \ldots, g(X_n)) = \det \frac{\partial r(f_\nu) - g(f_\nu)}{\partial X_\mu}(g(X_1), \ldots, g(X_n)) = \det(\frac{\partial m_\nu}{\partial X_\mu}(0))$ and $m_1(0) = \ldots = m_n(0) = 0$. If we write $m_i = \sum_{\nu \in \mathbb{N}_0^n} b_{i,\nu} X^\nu$ with $b_{i,\nu} \in B$ then, for every $i = 1, \ldots, n$, there exists a bounded subset K of B with $b_{i,\nu} \in K^{|\nu|}$ for any $\nu \in \mathbb{N}_0^n$. (That is true for any element of $B\langle X \rangle_{f(T)}$.) Let W be a neighbourhood of 0 in B such that $W \subseteq U$ and $w \cdot x$ is power-bounded in B for any $w \in W$ and $x \in \bigcup_{i=1}^n f(T_i)$. Then, for any $i = 1, \ldots, n$ and $x \in W^{\times n}$, the series $\sum_{\nu \in \mathbb{N}_0^n} b_{i,\nu} x^\nu$ converges to an element of B which we denote by $m_i(x)$. By (1) there exists a neighbourhood V of 0 in B with $V^{\times n} \subseteq \{(m_1(x), \ldots, m_n(x)) \mid x \in W^{\times n}\}$.

Let $(v_1, \ldots, v_n) \in V^{\times n}$ be given, and choose a $w = (w_1, \ldots, w_n) \in W^{\times n}$ with $v_i = m_i(w)$ for $i = 1, \ldots, n$. For any $i \in \{1, \ldots, n\}$ and $t \in T_i$, $f(t) \cdot (g(X_i) + w_i)$ is

power-bounded in B. Hence there exists a continuous A-algebra homomorphism $h: A\langle X\rangle_T \to B$ with $h(X_i) = g(X_i) + w_i$ for $i = 1, \ldots, n$. We have $h(f_i) = r(f_i)(g(X_1) + w_1, \ldots, g(X_n) + w_n) = (r(f_i) - g(f_i))(g(X_1) + w_1, \ldots, g(X_n) + w_n) + g(f_i) = s(r(f_i) - g(f_i))(w_1, \ldots, w_n) + g(f_i) = m_i(w) + g(f_i) = v_i + g(f_i)$.

(3) Let A be an affinoid ring, B a complete affinoid ring of topologically finite type over A, and $b_1, \ldots, b_n \in B^\flat$ a system of topological generators of B over A (i.e., there exist a quotient mapping $h: A\langle X_1, \ldots, X_n\rangle_{T_1, \ldots, T_n} \to B$ over A with $b_i = h(X_i)$ for $i = 1, \ldots, n$ ($T_1, \ldots, T_n$ are finite subsets of $A^\flat$ with $T_i \cdot A^\flat$ open in $A^\flat$)). Then there exists a neighbourhood U of 0 in $B^\flat$ such that every continuous ring homomorphism $f: B \to B$ over A with $f(b_i) \in b_i + U$ for $i = 1, \ldots, n$ is an isomorphism. More precisely, for every neighbourhood V of 0 in $B^\flat$, there exists a neighbourhood U of 0 in $B^\flat$ such that every continuous ring homomorphism $f: B \to B$ over A with $f(b_i) \in b_i + U$ for $i = 1, \ldots, n$ is an isomorphism with $f^{-1}(b_i) \in b_i + V$ for $i = 1, \ldots, n$.

Proof. The assertion is trivial if the topology of $B^\flat$ is discrete. Hence we may assume that $B^\flat$ is noetherian ((1.1.1)). Let a neighbourhood V of 0 in $B^\flat$ be given. Let A_0 be a ring of definition of $A^\flat$ with $A_0 \subseteq A^+$, and let $I \subseteq A_0$ be an ideal of definition of A_0. There exists a ring of definition B_0 of $B^\flat$ such that $A_0[t \cdot b_i \mid i = 1, \ldots, n, t \in T_i]$ is a dense subring of B_0, and B^+ is the integral closure of $A^+ \cdot B_0$ in $B^\flat$, and $B^\flat = A^\flat \cdot B_0[b_1, \ldots, b_n]$. We choose a $k \in \mathbb{N}$ such that $U := I^k \cdot B_0$ satisfies $U \subseteq V$ and $T_i \cdot U \subseteq I \cdot B_0$ for $i = 1, \ldots, n$. Let $f: B \to B$ be a continuous ring homomorphism over A with $f(b_i) \in b_i + U$ for $i = 1, \ldots, n$. We show that f is an isomorphism with $f^{-1}(b_i) \in b_i + U$ for $i = 1, \ldots, n$. For $i = 1, \ldots, n$ and $t \in T_i$, we have $f(t \cdot b_i) = t \cdot f(b_i) \subseteq t \cdot (b_i + U) \subseteq t \cdot b_i + t \cdot U \subseteq t \cdot b_i + I \cdot B_0$. Since $A_0[t \cdot b_i \mid i = 1, \ldots, n, t \in T_i]$ is dense in B_0, we obtain $f(b) \in b + I \cdot B_0$ for every $b \in B_0$. Hence $f(b) \in b + I^{n+1} \cdot B_0$ for every $n \in \mathbb{N}_0$ and every $b \in I^n \cdot B_0$, i.e. $f(I^n \cdot B_0) \subseteq I^n \cdot B_0$ and the induced mapping $I^n \cdot B_0 / I^{n+1} \cdot B_0 \to I^n \cdot B_0 / I^{n+1} \cdot B_0$ is the identity. We conclude from [B, III.2.9 Cor. 3] that $f(B_0) = B_0$ and hence $f(U) = U$. Since $f(b_i) \in b_i + U$, there exists a $c_i \in U$ with $f(b_i) = b_i + f(c_i)$, i.e. $b_i = f(b_i - c_i)$. This implies that $f: B^\flat \to B^\flat$ is surjective (since $B^\flat = A^\flat \cdot B_0[b_1, \ldots, b_n]$), and hence $f: B^\flat \to B^\flat$ is bijective (since $B^\flat$ is noetherian). Then $f(B^+) = B^+$, since B^+ is the integral closure of $A^+ \cdot B_0$ in $B^\flat$, and $f: B^\flat \to B^\flat$ is a homeomorphism, since the topology of B_0 is the $I \cdot B_0$-adic topology. Hence f is an isomorphism of affinoid rings.

(4) Let A be an affinoid ring, $T_1, \ldots, T_n$ finite subsets of $A^\flat$ with $T_i \cdot A^\flat$ open in $A^\flat$ for $i = 1, \ldots, n$, and let U be a rational subset of the affinoid adic space $X := \mathrm{Spa}A\langle X_1, \ldots, X_n\rangle_{T_1, \ldots, T_n}$. We consider X_i as an element of $\mathcal{O}_X(X)$ and hence we have $X_i|U \in \mathcal{O}_X(U)$ ($i = 1, \ldots, n$). There exists a neighbourhood H of 0 in $\mathcal{O}_X(U)$ such that every $\mathrm{Spa}A$-morphism $f: U \to X$ with $f^*(X_i) \in (X_i|U) + H$ for $i = 1, \ldots, n$ satisfies: $f(U) \subseteq U$ and $f: U \to U$ is an isomorphism of adic spaces. More precisely, for every neighbourhood G of 0 in $\mathcal{O}_X(U)$, there exists a neighbourhood H of 0 in $\mathcal{O}_X(U)$ such that every $\mathrm{Spa}A$-morphism $f: U \to X$ with $f^*(X_i) \in (X_i|U) + H$ for $i = 1, \ldots, n$ satisfies: $f(U) \subseteq U$ and $f: U \to U$ is an isomorphism with $(f^{-1})^*(X_i|U) \in (X_i|U) + G$ for $i = 1, \ldots, n$.

Proof. Let a neighbourhood G of 0 in $\mathcal{O}_X(U)$ be given. Let $s_0, s_1, \ldots, s_m$ be elements of $\mathcal{O}_X(X)$ such that $\{s_1, \ldots, s_m\} \cdot \mathcal{O}_X(X)$ is open in $\mathcal{O}_X(X)$ and $U = R(\frac{s_1, \ldots, s_m}{s_0})$. Then $\{s_1|U, \ldots, s_m|U\} \cdot \mathcal{O}_X(U)$ is open in $\mathcal{O}_X(U)$ and $U = \{u \in U \mid v_u(s_i|U) \leq v_u(s_0|U) \neq 0 \text{ for } i = 1, \ldots, m\}$. Then by [Hu$_1$, 3.10] there exists a neighbourhood H''' of 0 in $\mathcal{O}_X(U)$ such that, for $s_i' \in (s_i|U) + H'''(i = 1, \ldots, m)$, $U = \{u \in U \mid v_u(s_i') \leq v_u(s_0') \neq 0 \text{ for } i = 1, \ldots, m\}$. Let H'' be a neighbourhood of 0 in $\mathcal{O}_X(U)$ such that $f^*(s_i) \in (s_i|U) + H'''(i = 0, \ldots, m)$ for any Spa$A$-morphism $f: U \to X$ with $f^*(X_i) \in (X_i|U) + H''(i = 1, \ldots, n)$. Then $f(U) \subseteq U$ for any Spa$A$-morphism $f: U \to X$ with $f^*(X_i) \in (X_i|U) + H''$ for $i = 1, \ldots, n$. Let Q be a finite subset of $A^\flat$ such that $Q \cdot A^\flat$ is open in $A^\flat$ and $Q \cdot (s_0|U)^{-1} \in \mathcal{O}_X^+(U)$. Then we have the quotient mapping $q: (\mathcal{O}_X(X), \mathcal{O}_X^+(X))\langle X_{n+1}, \ldots, X_{n+m+1}\rangle_{\{1\}, \ldots, \{1\}, Q} \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ over $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ with $q(X_{n+i}) = \frac{s_i|U}{s_0|U}$ for $i = 1, \ldots, m$ and $q(X_{n+m+1}) = \frac{1}{s_0|U}$. Hence we have the quotient mapping $p: A\langle X_1, \ldots, X_{n+m+1}\rangle_{T_1, \ldots, T_n, \{1\}, \ldots, \{1\}, Q} \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ over A with $p(X_i) = X_i|U$ for $i = 1, \ldots, n$ and $p(X_i) = q(X_i)$ for $i = n+1, \ldots, n+m+1$. This shows that $X_1|U, \ldots, X_n|U, \frac{s_1|U}{s_0|U}, \ldots, \frac{s_m|U}{s_0|U}, \frac{1}{s_0|U}$ form a set of topological generators of $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ over A. Then by (3) there exists a neighbourhood H' of 0 in $\mathcal{O}_X(U)$ such that any Spa$A$-morphism $f: U \to U$ with $f^*(X_i|U) \in (X_i|U) + H'$ for $i = 1, \ldots, n$, $f^*(\frac{s_i|U}{s_0|U}) \in \frac{s_i|U}{s_0|U} + H'$ for $i = 1, \ldots, m$ and $f^*(\frac{1}{s_0|U}) \in \frac{1}{s_0|U} + H'$ is an isomorphism with $(f^{-1})^*(X_i|U) \in (X_i|U) + G$ for $i = 1, \ldots, n$. Let H be a neighbourhood of 0 in $H' \cap H''$ such that any Spa$A$-morphism $f: U \to U$ with $f^*(X_i|U) \in (X_i|U) + H$ (for $i = 1, \ldots, n$) satisfies $f^*(\frac{s_i|U}{s_0|U}) \in \frac{s_i|U}{s_0|U} + H'$ for $i = 1, \ldots, m$ and $f^*(\frac{1}{s_0|U}) \in \frac{1}{s_0|U} + H'$. Then any Spa$A$-morphism $f: U \to X$ with $f^*(X_i) \in (X_i|U) + H$ for $i = 1, \ldots, n$ satisfies: $f(U) \subseteq U$ and $f: U \to U$ is an isomorphism with $(f^{-1})^*(X_i|U) \in (X_i|U) + G$ for $i = 1, \ldots, n$.

After this preparations we can prove that (ii) implies (iii). We use the notations introduced in (ii). Put $Z := \text{Spa}B$ and $\Delta := \det(\frac{\partial f_i}{\partial X_j})_{i,j=1, \ldots, n} \in B^\flat = \mathcal{O}_Z(Z)$. Since $\Delta(x) \neq 0$ for any $x \in \text{Spa}B/I$, there exist $s_1, \ldots, s_m \in A^\flat$ such that $\{s_1, \ldots, s_m\} \cdot A^\flat$ is open in $A^\flat$ and $v_x(s_i) \leq v_x(\Delta) \neq 0$ for any $x \in \text{Spa}B/I$, $i = 1, \ldots, m$ ([Hu$_1$, 3.11]). Let U denote the rational subset $\{x \in Z \mid v_x(s_i) \leq v_x(\Delta) \neq 0 \text{ for } i = 1, \ldots, m\}$ of Z. Let H be a neighbourhood of 0 in $\mathcal{O}_Z(U)$ such that $t(h + X_i|U) \in \mathcal{O}_Z^+(U)$ for every $h \in H$, $i \in \{1, \ldots, n\}$, $t \in T_i$. Then, for any $h = (h_1, \ldots, h_n) \in H^{\times n}$, we have a Y-morphism $f_h: U \to Z$ with $f_h^*(X_i) = h_i + X_i|U$ for $i = 1, \ldots, n$ ([Hu, 3.5 i]). Decreasing H, we may assume by (4) that $f_h(U) \subseteq U$ and $f_h: U \to U$ is an isomorphism for any $h \in H^{\times n}$. Since $\Delta|U$ is a unit of $\mathcal{O}_Z(U)$ there exists by (2) a neighbourhood V of 0 in $\mathcal{O}_Z(U)$ such that, for every $(v_1, \ldots, v_n) \in V^{\times n}$, there exists a $h \in H^{\times n}$ with $f_h^*(f_i) = v_i + f_i|U$ for $i = 1, \ldots, n$. Since $Z - U$ is quasi-compact and, for every $x \in Z - U$, there exists a $i \in \{1, \ldots, n\}$ with $f_i(x) \neq 0$, there exists a neighbourhod W of 0 in $B^\flat$ such that, for every $x \in Z - U$ and every $(w_1, \ldots, w_n) \in W^{\times n}$, there exists a $i \in \{1, \ldots, n\}$ with $(w_i + f_i)(x) \neq 0$. Let S be a neighbourhood of 0 in $B^\flat$ such that $S \subseteq W$ and $\{s|U \mid s \in S\} \subseteq V$. Then, for any $(s_1, \ldots, s_n) \in S^{\times n}$, if we put $I' := \{f_1 + s_1, \ldots, f_n + s_n\} \cdot B^\flat$ then SpaB/I'

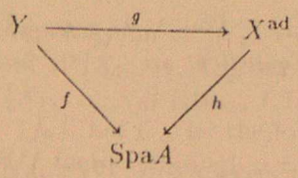
is Y-isomorphic to SpaB/I and the image of $\det(\frac{\partial(f_i+s_i)}{\partial X_j})_{i,j=1,\dots,n} \in B^\flat$ in $B^\flat/I'$ is an unit of $B^\flat/I'$. Indeed, SpaB/I and SpaB/I' are contained in U, and there exists a $h \in H^{\times n}$ such that $f_h: U \xrightarrow{\sim} U$ induces an isomorphism between the adic subspaces SpaB/I' and SpaB/I of U. Since the image of $A^\flat[X_1, \dots, X_n]$ in $B^\flat$ is dense in $B^\flat$, we obtain (iii).

(iii) $\Longrightarrow$ (i) follows from (1.6.7.i) and (1.6.9).

Corollary 1.7.2. i) *Let A be an affinoid ring and $T_1, \dots, T_n$ finite subsets of $A^\flat$ such that $T_i \cdot A^\flat$ is open in $A^\flat$ for $i = 1, \dots, n$. Let B be the affinoid ring $A\langle X_1, \dots, X_n\rangle_{T_1,\dots,T_n}$ and let $f_1, \dots, f_n$ be elements of $B^\flat$ such that the image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\dots,n} \in B^\flat$ in $B^\flat/I$ is a unit of $B^\flat/I$ where $I := f_1 B^\flat + \dots + f_n B^\flat$. Let V be a neighbourhood of 0 in $B^\flat/I$. Then there exists a neighbourhood S of 0 in $B^\flat$ such that if $f_1' \in f_1 + S, \dots, f_n' + S$ and $I' := f_1' B^\flat + \dots + f_n' B^\flat$ then the image of $\det(\frac{\partial f_i'}{\partial X_j})_{i,j=1,\dots,n} \in B^\flat$ in $B^\flat/I'$ is a unit of $B^\flat/I'$ and there exists a Spa$A$-isomorphism c: Spa$B/I \to$ SpaB/I' with $c^*(\overline{T}_i') \in \overline{T}_i + V$ for $i = 1, \dots, n$ where $\overline{T}_i \in B^\flat/I$ (resp. $\overline{T}_i' \in B^\flat/I'$) is the image of $T_i \in B^\flat$ in $B^\flat/I$ (resp. $B^\flat/I'$).*
ii) *Let $X \to Y =$ Spa$A$ be an étale morphism of affinoid adic spaces. If A is Tate then we have a description of X as in (1.7.1.iii) with $T_1 = \dots = T_n = \{1\}$.*

Proof. i) We use the notations from the end of the proof of (1.7.1). We consider SpaB/I as an adic subspace of U. Let G be a neighbourhood of 0 in $\mathcal{O}_Z(U)$ such that, for any $g \in G$, the restriction $\overline{g} \in B^\flat/I$ of g to SpaB/I is contained in V. Decreasing H, we may assume by (4) in the proof of (1.7.1) that, for any $h \in H^{\times n}$, $f_h(U) \subseteq U$ and $f_h: U \to U$ is an isomorphism with $(f_h^{-1})^*(X_i|U) \in (X_i|U) + G$ for $i = 1, \dots, n$. Then the arguments at the end of the proof of (1.7.1) show that the isomorphisms $f_h^{-1}: U \to U$ induce the isomorphism c: Spa$B/I \to$ SpaB/I' we are looking for.
ii) If an affinoid ring A is Tate then every affinoid ring of topologically finite type over A is a quotient of some $A\langle X_1, \dots, X_n\rangle$ ([Hu, 3.5.iii]). Therefore ii) follows from i) and the proof of (1.7.1.i) $\Rightarrow$ (1.7.1.ii). □

Corollary 1.7.3. *Let A be an affinoid ring, D a ring and $\varphi: D \to A^\flat$ a ring homomorphism. φ induces a morphism of locally ringed spaces* Spa$A \to$ Spec$D$. *Then by* [Hu, 3.8] *every scheme X locally of finite type over* Spec$D$ *defines an adic space $X^{\mathrm{ad}} := X \times_{\mathrm{Spec}D}$ Spa$A$ locally of finite type over* Spa$A$.
i) *If $X \to$ Spec$D$ is an étale morphism of schemes then $X^{\mathrm{ad}} \to$ Spa$A$ is an étale morphism of adic spaces.*
ii) *Let Y be an affinoid adic space and $f: Y \to$ Spa$A$ an étale morphism. If $\mathrm{im}(\varphi)$ is dense in $A^\flat$ then there exist an affine scheme X, an étale morphism $X \to$ Spec$D$ and an open embedding of adic spaces $g: Y \to X^{\mathrm{ad}}$ such that the diagram*



commutes where h is the structure morphism of X^{ad}.

iii) *Let Y be an affinoid adic space and $f: Y \to \mathrm{Spa}A$ a morphism of adic spaces. Then equivalent are*

a) *f is étale*

b) *There exist an affinoid ring B and a ring homomorphism of algebraically finite type $A \to B$ between affinoid rings ((1.2.5)) such that $A^\flat \to B^\flat$ is étale (in the algebraic sense) and Y is $\mathrm{Spa}A$-isomorphic to $\mathrm{Spa}B$.*

Proof. i) We may assume that X is affine. Then there exist $f_1, \ldots, f_n \in A^\flat[X_1, \ldots, X_n] =: C$ such that X is $\mathrm{Spec}A^\flat$-isomorphic to $\mathrm{Spec}C/(f_1, \ldots, f_n)$ and the image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n}$ in $C/(f_1, \ldots, f_n)$ is an unit of $C/(f_1, \ldots, f_n)$. Let $x \in X^{ad}$ be given. Then there exist an open neighbourhood U of x in X^{ad} and finite subsets $T_1, \ldots, T_n$ of $A^\flat$ with $T_i \cdot A^\flat$ open in $A^\flat$ for $i = 1, \ldots, n$ such that U is $\mathrm{Spa}A$-isomorphic to $\mathrm{Spa}B/I$ with $B = A\langle X_1, \ldots, X_n\rangle_{T_1, \ldots, T_n}$ and $I = f_1 B^\flat + \ldots + f_n B^\flat$. Hence by (1.7.1) U is étale over $\mathrm{Spa}A$.

ii) By (1.7.1) and (1.7.2.i) there exist finite subsets $T_1, \ldots, T_n$ of $A^\flat$ with $T_i \cdot A^\flat$ open in $A^\flat$ for $i = 1, \ldots, n$ and $f_1, \ldots, f_n \in D[X_1, \ldots, X_n]$ such that Y is $\mathrm{Spa}A$-isomorphic to $\mathrm{Spa}B/I$ with $B := A\langle X_1, \ldots, X_n\rangle_{T_1, \ldots, T_n}$, $I := f_1 B^\flat + \ldots + f_n B^\flat$, and the image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n}$ in $B^\flat/I$ is an unit of $B^\flat/I$. Let Z be the $\mathrm{Spec}D$-scheme $\mathrm{Spec}D[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$. Then Y can be considered as an open subspace of Z^{ad}. Put $\Delta := \det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n}$ and $X := \{z \in Z \mid \Delta \text{ is an unit in } \mathcal{O}_{Z,z}\}$. Then X is an open affine subscheme of Z which is etale over $\mathrm{Spec}D$, and Y is an open subspace of X^{ad}.

iii) It is easily seen that if b) is satisfied then f has a factorization as in ii) and hence f is étale. Now we assume a) and show b). By (1.7.1) there exist finite subsets $T_1, \ldots, T_n$ of $A^\flat$ with $T_i \cdot A^\flat$ open in $A^\flat$ for $i = 1, \ldots, n$ and $f_1, \ldots, f_n \in A^\flat[X_1, \ldots, X_n]$ such that Y is $\mathrm{Spa}A$-isomorphic to $\mathrm{Spa}C$ with $C := A\langle X_1, \ldots, X_n\rangle_{T_1, \ldots, T_n}/(f_1, \ldots, f_n)$ and the image c of $d := \det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n}$ in $C^\flat$ is a unit of $C^\flat$. Since $c(x) \neq 0$ for every $x \in \mathrm{Spa}C$, there exists a finite subset T of $A^\flat$ such that $T \cdot A^\flat$ is open in $A^\flat$ (and then also $T \cdot C^\flat$ is open in $C^\flat$) and $|m(x)| \leq |c(x)| \neq 0$ for every $x \in \mathrm{Spa}C$, $m \in T$. Then $C' := C\langle \frac{T}{c} \rangle = C\langle X \rangle_T/(1 - cX) = A\langle X_1, \ldots, X_n, X\rangle_{T_1, \ldots, T_n, T}/(f_1, \ldots, f_n, 1 - dX)$. Obviously there exists a ring homomorphism of algebraically finite type $A \to B$ between affinoid rings such that $A^\flat \to B^\flat$ is étale and C is A-isomorphic to the completion of B. □

(1.6.8) and (1.7.3.iii) imply

Corollary 1.7.4. *Every unramified morphism of adic spaces is locally quasi-finite.*

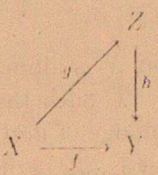
Proposition 1.7.5. *Let $f: X \to Y$ be a morphism of analytic adic spaces which is locally of finite type. Equivalent are*

a) *f is étale.*

b) *f is flat (i.e., for every $x \in X$, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$) and unramified.*

c) *For every maximal point x of X, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is a finite separable extension field of $k(f(x))$.*

Proof. a) $\Rightarrow$ b) by the subsequent Lemma (1.7.6).
b) $\Rightarrow$ a): We may assume that X and Y are affinoid. Then by (1.6.8) there exists a commutative diagram



where g is a closed embedding and h is étale. In order to show that f is étale it suffices to show that g is an open embedding. For that it suffices to show that g is flat.
Let $s: X \to X \times_Y Z$ be the graph morphism of g and $p: X \times_Y Z \to Z$ the projection. Since $g = p \circ s$, it suffices to show that p and s are flat. By (1.7.4) f is quasi-finite. Hence by (1.6.4.ii) p is flat. s is an open embedding, since s is the pull back of the diagonal morphism $\Delta: Z \to Z \times_Y Z$ by $g \times \mathrm{id}: X \times_Y Z \to Z \times_Y Z$ and Δ is an open embedding by (1.6.8).
c) $\Rightarrow$ b): f is flat by (1.1.10.iii). By (1.5.2) f is locally quasi finite. Then by (1.6.4.i) and Nakayama's Lemma, $(\Omega_{X|Y})_x = 0$ for every maximal point x of X. Then $\Omega_{X|Y} = 0$, and hence by (1.6.8) f is unramified.
b) $\Rightarrow$ c): By (1.7.4) f is quasi-finite. Then, for every maximal point x of X and $A := \mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$, A is finite over $k(f(x))$ and $\Omega_{A|k(f(x))} = 0$ ((1.6.8) and (1.6.4)). Hence A is a finite separable extension field of $k(f(x))$. □

Lemma 1.7.6. *Let $X \to Y$ be an étale morphism between affinoid adic spaces. Then the ring homomorphism $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is flat.*

Proof. Let A be the affinoid ring $(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$. By (1.7.1) we may assume that there exist finite subsets $T_1, \ldots, T_n$ of $A^\flat$ with $T_i \cdot A^\flat$ open in $A^\flat$ for $i = 1, \ldots, n$ and $f_1, \ldots, f_n \in A^\flat[X_1, \ldots, X_n]$ such that if we put $B = A^\flat\langle X_1, \ldots, X_n\rangle_{T_1, \ldots, T_n}$ and $I = f_1 B + \ldots + f_n B$ then the image of $\Delta = \det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n} \in A^\flat[X_1, \ldots, X_n]$ in B/I is a unit of B/I and $\mathcal{O}_X(X) = B/I$.
We distinguish two cases.
1) Assume that $A^\flat$ has a noetherian ring of definition.

Then there exists a f-adic topology on $A^\flat[X_1, \ldots, X_n]$ such that B is the completion of $A^\flat[X_1, \ldots, X_n]$ and $A^\flat[X_1, \ldots, X_n]$ has a noetherian ring of definition. Therefore B is flat over $A^\flat[X_1, \ldots, X_n]$ ([Hu$_1$, 1.7.i]), and hence B/I is flat over $C := A^\flat[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$. Let C_Δ be the localization of C by Δ. Then the ring homomorphism $C \to B/I$ factors through a ring homomorphism $C_\Delta \to B/I$ which is also flat. Since C_Δ is étale over $A^\flat$ and therefore flat over $A^\flat$, we obtain that B/I is flat over $A^\flat$.

ii) Assume that $A^\flat$ is a strongly noetherian Tate ring.

Then by (1.7.2.ii) we may assume $T_1 = \ldots = T_n = \{1\}$. We show that $A^\flat\langle X_1, \ldots, X_n\rangle$ is flat over $A^\flat[X_1, \ldots, X_n]$. (Then we can proceed as in (i)). Using induction on n, we may assume $n = 1$. Every $A^\flat$-module homomorphism $M \to N$ between finitely generated $A^\flat$-modules is strict if we equip M and N with its natural $A^\flat$-module topologies ([Hu, 2.4]). This implies that $A^\flat\langle X\rangle$ is flat over $A^\flat$, cf. [FP, III.7.8, 7.9]. Since also $A^\flat[X]$ is flat over $A^\flat$, it is enough to prove that, for every residue field k of $A^\flat$, the ring homomorphism $A^\flat[X] \otimes_{A^\flat} k \to A^\flat\langle X\rangle \otimes_{A^\flat} k$ is flat ([EGA, 0_{III}.10.2.5]). Let $\mathfrak{p}$ be a prime ideal of $A^\flat$. Since the ideal $\mathfrak{p} \cdot A^\flat\langle X\rangle$ is closed in $A^\flat\langle X\rangle$, we have $\mathfrak{p} \cdot A^\flat\langle X\rangle = \{\sum_{n \in \mathbb{N}_0} a_n X^n \in A^\flat\langle X\rangle \mid a_n \in \mathfrak{p} \text{ for every } n \in \mathbb{N}_0\}$. This shows that $\mathfrak{p} \cdot A^\flat\langle X\rangle$ is a prime ideal of $A^\flat\langle X\rangle$. Then $M := A^\flat\langle X\rangle/\mathfrak{p} \cdot A^\flat\langle X\rangle$ is a torsion-free C-module with $C := A^\flat[X]/\mathfrak{p} \cdot A^\flat[X]$. Let S be the multiplicative system $A^\flat - \mathfrak{p}$. Then the localization M_S is a torsion-free C_S-module. Since C_S is a Dedekind ring, we obtain that M_S is flat over C_S. □

(1.7.7) Let A be an affinoid ring and $f: Y \to \mathrm{Spa}A$ a smooth morphism of adic spaces. Then, for every $y \in Y$, there exist an open affinoid neighbourhood U of y in Y, an affinoid ring B and a ring homomorphism of algebraically finite type $A \to B$ between affinoid rings such that U is $\mathrm{Spa}A$-isomorphic to $\mathrm{Spa}B$ and the morphism of schemes $\mathrm{Spec}B^\flat \to \mathrm{Spec}A^\flat$ is smooth ((1.6.10), (1.7.3.iii)).

Proposition 1.7.8. *Every smooth morphism of adic spaces is open.*

Proof. The assertion follows from (1.7.7) and the subsequent Lemma (1.7.9). □

Lemma 1.7.9. *Let $f: A \to B$ be an adic morphism of affinoid rings such that the ring homomorphism $f: A^\flat \to B^\flat$ is flat and finitely presented and there exists a finite subset L of B^+ such that B^+ is the integral closure of $A^+[L]$ in $B^\flat$. Then $\mathrm{Spa}(f): \mathrm{Spa}B \to \mathrm{Spa}A$ is an open mapping.*

Proof. Put $\varphi := \mathrm{Spa}(f)$. Let $U = R(\frac{T}{s})$ be a rational subset of $\mathrm{Spa}B$. We have to show that $\varphi(U)$ is open in $\mathrm{Spa}A$. Replacing B by the affinoid ring $B(\frac{T}{s})$ ([Hu, §1]), we may assume $U = \mathrm{Spa}B$.

Put $X_A := \{v \in \mathrm{Spv}A^\flat \mid v(a) \leq 1$ for every $a \in A^+$ and $v(a) < 1$ for every $a \in (A^\flat)^{\circ\circ}\} \subseteq \mathrm{Spv}A^\flat$ and $X_B := \{v \in \mathrm{Spv}B^\flat \mid v(b) \leq 1$ for every $b \in A^+[L]$ and $v(b) < 1$ for every $b \in (A^\flat)^{\circ\circ}\} = \{v \in \mathrm{Spv}B^\flat \mid v(b) \leq 1$ for every $b \in B^+$ and $v(b) < 1$ for every $b \in (B^\flat)^{\circ\circ}\} \subseteq \mathrm{Spv}B$, and let $\psi: X_B \to X_A$ be the restriction of $\mathrm{Spv}(f): \mathrm{Spv}B^\flat \to \mathrm{Spv}A^\flat$. By [Hu$_1$, 2.6.iii] we have retractions $r_A: \mathrm{Spv}A \to$

$\mathrm{Spv}(A^\flat, (A^\flat)^{00} \cdot A^\flat)$ and $r_B : \mathrm{Spv}B \to \mathrm{Spv}(B^\flat, (B^\flat)^{00} \cdot B^\flat)$ which give via restrictions retractions $s_A : X_A \to \mathrm{Spa}A$ and $s_B : X_B \to \mathrm{Spa}B$ ([Hu$_1$, 3.1]). The diagram

(*)
$$\begin{array}{ccccc} \mathrm{Spa}B & \hookrightarrow & X_B & \xrightarrow{s_B} & \mathrm{Spa}B \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Spa}A & \hookrightarrow & X_A & \xrightarrow{s_A} & \mathrm{Spa}A \end{array}$$
(with $\mathrm{id}_{\mathrm{Spa}B}$ over the top row and $\mathrm{id}_{\mathrm{Spa}A}$ under the bottom row)

commutes. By [HK, 2.1.4] $\mathrm{Spv}(f) : \mathrm{Spv}B^\flat \to \mathrm{Spv}A^\flat$ is open. Since X_B is open in $\mathrm{Spv}(f)^{-1}(X_A)$, we obtain that $\mathrm{im}(\psi)$ is open in X_A. The commutative diagram (*) gives $\mathrm{im}(\varphi) = \mathrm{im}(\psi) \cap \mathrm{Spa}A$. Hence $\mathrm{im}(\varphi)$ is open in $\mathrm{Spa}A$. □

Definition 1.7.10. Let $f : X \to Y$ be a morphism of rigid analytic varieties.
i) f is called *étale* if, for every $x \in X$, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is a finite separable extension field of the residue class field $\mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}$ ([SS, §1]).
ii) f is called *smooth* if, for every $x \in X$, there exists a commutative diagram

$$\begin{array}{ccc} U & & \\ \downarrow{\scriptstyle f} & \searrow^{g} & \\ V & \swarrow_{h} & \mathrm{Sp}\mathcal{O}_Y(V)\langle T_1, \ldots, T_n\rangle \end{array}$$

where U and V are affinoid admissible open neighbourhoods of x and $f(x)$ in X and Y with $f(U) \subseteq V$, h is the natural morphism and g is étale.

Proposition 1.7.11. *Let $f : X \to Y$ be a morphism of rigid analytic varieties and let $r(f) : r(X) \to r(Y)$ be the associated morphism of adic spaces. Then*
i) f is étale if and only if $r(f)$ is étale.
ii) f is smooth if and only if $r(f)$ is smooth.

Proof. i) If $r(f)$ is étale then f is étale by (1.7.5.a), c)). Assume that f is étale. Then, for any affinoid admissible open subsets U and V of X and Y with $f(U) \subseteq V$, $\mathcal{O}_X(U)$ is flat over $\mathcal{O}_Y(V)$. Hence, for every $x \in r(X)$ and $y = r(f)(x) \in r(Y)$, $\mathcal{O}_{r(X),x}$ is flat over $\mathcal{O}_{r(Y),y}$. By (1.5.7) $r(f)$ is locally quasi-finite. Then by (1.6.4.i) and Nakayama's Lemma, $(\Omega_{r(X)|r(Y)})_x = 0$ for every $x \in X \subseteq r(X)$. This implies $\Omega_{r(X)|r(Y)} = 0$. Hence $r(f)$ is unramified ((1.6.8)). Now (1.7.5.a), b)) gives that $r(f)$ is étale.
ii) If $r(f)$ is smooth then f is smooth by (1.6.10). Assume that f is smooth.

Then $r(f)$ is smooth at every $x \in X \subseteq r(X)$ ((1.6.10)). Hence $\Omega_{r(X)|r(Y)}$ is a locally free $\mathcal{O}_{r(X)}$-module ((1.6.9.i)). Let $x \in r(X)$ be given. We show that $r(f)$ is smooth at x. We may assume that Y is affinoid. Choose an open affinoid neighbourhood U of x in $r(X)$ and $s_1, \ldots, s_n \in \mathcal{O}_{r(X)}(U)$ such that $\Omega_{r(X)|r(Y)}|U$ is a free $(\mathcal{O}_{r(X)}|U)$-module with basis $d_{r(X)|r(Y)}(s_1), \ldots, d_{r(X)|r(Y)}(s_n) \in \Omega_{r(X)|r(Y)}(U)$. Multiplying $s_1, \ldots, s_n$ with a suitable element of the ground field, we may assume that $s_1, \ldots, s_n \in \mathcal{O}^+_{r(X)}(U)$. Then there exists a commutative diagram

$$\begin{array}{ccc} U & & \\ \downarrow{\scriptstyle r(f)} & \searrow{\scriptstyle g} & \\ & & \mathrm{Spa}(\mathcal{O}_Y(Y), \mathcal{O}^\circ_Y(Y))\langle T_1, \ldots, T_n\rangle \\ & \swarrow{\scriptstyle h} & \\ r(Y) = \mathrm{Spa}(\mathcal{O}_Y(Y), \mathcal{O}^\circ_Y(Y)) & & \end{array}$$

where h is the natural morphism and $g^*(T_i) = s_i$ for $i = 1, \ldots, n$ ([Hu, 3.5.i]). Since $r(f)$ is smooth at every $q \in X \subseteq r(X)$, g is étale at every $q \in X \cap U$ ((1.6.9.iii)). Then g is étale by i). Hence $r(f)$ is smooth at x. Since x was an arbitrary point of $r(X)$, we obtain that $r(f)$ is smooth ((1.6.7.v)). □

1.8 Dimensions

For a scheme X of finite type over a field k, one can define $\dim X$ as the maximum of the lengths of the chains of irreducible closed subsets of X, or one can define $\dim X$ as the maximum of the transcendence degrees of the residue class fields of X over k. We will use these two concepts in order to define dimensions of adic spaces.

Definition 1.8.1. i) The dimension $\dim X \in \mathbb{N}_0 \cup \{+\infty, -\infty\}$ of a locally spectral topological space X is the supremum of the lengths n of the chains of specializations $x_0 \succ x_1 \succ \ldots \succ x_n$ of points of X. X is called of pure dimension n if every nonempty open subset U of X has dimension n.
ii) The dimension $\dim X/Y = \dim f$ of a spectral mapping $f: X \to Y$ between locally spectral topological spaces is the supremum of the dimensions of the fibres of f.

$$\dim X/Y = \dim f = \sup\{\dim f^{-1}(y) \mid y \in Y\} \in \mathbb{N}_0 \cup \{+\infty, -\infty\}.$$

f is called of pure dimension n if every nonempty fibre $f^{-1}(y)$ is of pure dimension n.

In particular, (1.8.1) defines the dimensions of adic spaces and of adic morphisms between adic spaces.

Definition 1.8.2. Let E be a subfield of a non-archimedean field F.
i) $E(F)$ denotes the topological closure in F of the algebraic closure of E in F.
ii) We define a sequence $E(F,1) \subseteq E(F,2) \subseteq \dots$ of subfields of F by $E(F,1) := E(F)$ and $E(F,n+1) := E(F,n)(F)$.
iii) $\mathrm{tr.c}(F|E) := \min\{\mathrm{trdeg}(L|E) \mid L$ a subfield of F with $E \subseteq L$ and $L(n,F) = F$ for some $n \in \mathbb{N}\} \in \mathbb{N}_0 \cup \{\infty\}$.

Remark 1.8.3. i) Let $D \to E \to F$ be continuous ring homomorphisms between non-archimedean fields. Then $\mathrm{tr.c}(F|D) \leq \mathrm{tr.c}(F|E) + \mathrm{tr.c}(E|D)$.
ii) Let

$$\begin{array}{ccc} F & \longleftarrow & D \\ \uparrow & & \uparrow \\ E & \longleftarrow & C \end{array}$$

be a commutative diagram of continuous ring homomorphisms between non-archimedean fields such that the subfield of F generated by E and D is dense in F. Then $\mathrm{tr.c}(F|E) \leq \mathrm{tr.c}(D|C)$.

Definition 1.8.4. Let $f: X \to Y$ be a morphism between analytic adic spaces. Put

$$\dim.\mathrm{tr} f^{-1}(y) := \sup\{\mathrm{tr}\ c(k(x)^\wedge|k(y)^\wedge) \mid x \in f^{-1}(y)\} \in \mathbb{N}_0 \cup \{+\infty, -\infty\} (y \in Y)$$
$$\dim.\mathrm{tr}(X/Y) = \dim.\mathrm{tr} f := \sup\{\dim.\mathrm{tr} f^{-1}(y) \mid y \in Y\} \in \mathbb{N}_0 \cup \{+\infty, -\infty\}.$$

Lemma 1.8.5. *Let $f: X \to Y$ be a morphism of analytic adic spaces.*
i) *For every $y \in Y$, $\dim f^{-1}(y) \leq \dim.\mathrm{tr} f^{-1}(y)$. Hence $\dim f \leq \dim.\mathrm{tr} f$.*
ii) *Let y_0, y_1 be points of Y with $y_0 \succ y_1$. Then $\dim.\mathrm{tr} f^{-1}(y_0) \geq \dim.\mathrm{tr} f^{-1}(y_1)$.*

Proof. i) Let x be a point of $f^{-1}(y)$, and let Γ_x and Γ_y be the value groups of the valuations v_x and v_y. By [B, VI.10.3 Cor.1] we have $\dim_{\mathbb{Q}}(\Gamma_x/\Gamma_y) \otimes \mathbb{Q} \leq \mathrm{tr.c}(k(x)^\wedge|k(y)^\wedge) \leq \dim.\mathrm{tr} f^{-1}(y)$. Hence, for every chain of specializations $x_0 \succ x_1 \succ \dots \succ x_m = x$ of points of $f^{-1}(y)$, we have $m \leq \dim.\mathrm{tr} f^{-1}(y)$. Hence $\dim f^{-1}(y) \leq \dim.\mathrm{tr} f^{-1}(y)$.
ii) For every $x_1 \in f^{-1}(y_1)$, there exists a $x_0 \in f^{-1}(y_0)$ with $x_0 \succ x_1$ ((1.1.10.v)). Since $k(y_1)^\wedge \xrightarrow{\sim} k(y_0)^\wedge$ and $k(x_1)^\wedge \xrightarrow{\sim} k(x_0)^\wedge$ ((1.1.10.iii)), we have $\mathrm{tr.c}(k(x_1)^\wedge|k(y_1)^\wedge) = \mathrm{tr.c}(k(x_0)^\wedge|k(y_0)^\wedge)$. Therefore $\dim.\mathrm{tr} f^{-1}(y_1) \leq \dim.\mathrm{tr} f^{-1}(y_0)$. □

Lemma 1.8.6. *Let $f: X \to S$ be a morphism of affinoid adic spaces where S is the spectrum of an analytic affinoid field. Put $n := \dim \mathcal{O}_X(X)$, and let s be a point of S.*
i) *Assume that $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ is of topologically finite type over $(\mathcal{O}_S(S), \mathcal{O}_S^+(S))$. Then $\dim f^{-1}(s) = \dim.\mathrm{tr} f^{-1}(s) \leq n$, and if $\mathcal{O}_X(X)$ is equidimensional and*

(iii) $f(X_{\mathrm{na}}) \subseteq Y_{\mathrm{na}}$.

In order to define morphisms of finite type between adic spaces we first introduce ring homomorphisms of topologically finite type between f-adic rings and affinoid rings. Let A be a f-adic ring and let $M_1,\dots,M_n$ be finite subsets of A such that $M_i \cdot A$ is open in A for every $i = 1,\dots,n$. We put

$$A\langle X\rangle_M = A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n} := \{\sum a_\nu X^\nu \in \hat{A}[[X_1,\dots,X_n]] \mid \text{for every neighbourhood } U \text{ of } 0 \text{ in } \hat{A}, a_\nu \notin M_1^{\nu_1}\cdot\ldots\cdot M_n^{\nu_n}\cdot U \text{ for only finitely many } \nu = (\nu_1,\dots,\nu_n) \in \mathbb{N}_0^n\}.$$

Obviously $A\langle X\rangle_M$ is a subgroup of $\hat{A}[[X_1,\dots,X_n]]$. We equip $A\langle X\rangle_M$ with the group topology such that the sets $\{\sum a_\nu X^\nu \in A\langle X\rangle_M \mid a_\nu \in M_1^{\nu_1}\cdot\ldots\cdot M_n^{\nu_n}\cdot U$ for every $\nu = (\nu_1,\dots,\nu_n) \in \mathbb{N}_0^n\}$ (U a neighbourhood of 0 in $\hat{A}$) from a fundamental system of neighbourhoods of 0. Using (1.1), one can easily see that $A\langle X\rangle_M$ is a subring of $\hat{A}[[X_1,\dots,X_n]]$, and moreover $A\langle X\rangle_M$ is a complete f-adic ring. The natural ring homomorphism $A \to A\langle X\rangle_M$ is continuous.

We put

$$A\langle X\rangle = A\langle X_1,\dots,X_n\rangle := A\langle X_1,\dots,X_n\rangle_{\{1\},\dots,\{1\}}.$$

We say that a ring homomorphism $f: A \to B$ from A to a complete f-adic ring B is of **topologically finite type** if there exist $n \in \mathbb{N}_0$, finite subsets $M_1,\dots,M_n$ of A with $M_i \cdot A$ open in A for $i = 1,\dots,n$ and a surjective, continuous, open ring homomorphism $g: A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n} \to B$ such that $f = g \circ h$, where $h: A \to A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n}$ is the natural ring homomorphism. One can easily check

Lemma 3.3 (i) (Universal property of $A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n}$) *Let $h: A \to A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n}$ be the natural ring homomorphism. Then $h(m)X_i$ is power-bounded in $A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n}$ for every $i \in \{1,\dots,n\}$ and $m \in M_i$, and if $f: A \to B$ is a continuous ring homomorphism from A to a complete* f-adic *ring B and $b_1\dots,b_n$ are elements of B such that $f(m)b_i$ is power-bounded in B for every $i \in \{1,\dots,n\}$ and $m \in M_i$ then there exists a unique continuous ring homomorphism $g: A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n} \to B$ with $f = g\circ h$ and $b_i = g(X_i)$ for $i = 1,\dots,n$.*

(ii) *Let $f: A \to B$ be a ring homomorphism from A to a complete* f-adic *ring B. Then the following conditions are equivalent*

(a) *f is of topologically finite type*

(b) *f is adic, there exists a finite subset M of B such that $A[M]$ is dense in B, and there exist rings of definition A_0, B_0 of A, B and a finite subset N of B_0 such that $f(A_0) \subseteq B_0$ and $A_0[N]$ is dense in B_0.*

(iii) *If A is Tate then a ring homomorphism $f: A \to B$ from A to a complete* f-adic *ring B is of topologically finite type if and only if f factors through a surjective, continuous and open ring homomorphism $g: A\langle X_1,\dots,X_n\rangle \to B$ for some $n \in \mathbb{N}_0$.*

(iv) *Let B, C be complete* f-adic *rings and $f: A \to B$ and $g: B \to C$ continuous ring homomorphisms. If f and g are of topologically finite type then $g \circ f$ is of topologically finite type, and if $g \circ f$ is of topologically finite type then g is of topologically finite type.*

Corollary 3.4 (i) *Let $s_1,\dots,s_n$ be elements of A and $T_1,\dots,T_n$ finite subsets of A such that $T_i \cdot A$ is open for $i = 1,\dots,n$. Then the natural ring homomorphism $A \to A\left\langle \frac{T_1}{s_1},\dots,\frac{T_n}{s_n}\right\rangle$ is of topologically finite type.*
(ii) *If A is Tate then the following conditions are equivalent*

(a) *A is strongly noetherian.*
(b) *Every Tate ring of topologically finite type over A is noetherian.*
(c) *Every Tate ring of topologically finite type over A is strongly noetherian.*

(iii) *Let B be a complete f-adic ring of topologically finite type over A. If A has a noetherian ring of definition then also B has a noetherian ring of definition.*

Proof. Applying (3.3.ii), (i) follows immediately from the definition of $A\left\langle \frac{T_1}{s_1},\dots,\frac{T_n}{s_n}\right\rangle$. (ii) follows from (3.3.iii,iv), and (iii) follows from (3.3.ii). □

Let k be a complete, non-archimedean, valued field. Then k is a Tate ring. In rigid analytic geometry one calls the complete topological k-algebras which are quotients of some $k\langle X_1,\dots,X_n\rangle$ affinoid algebra or Tate algebras over k. We call them, according to (3.3.iii), Tate rings of topologically finite type over k.

Let $A = (A^\triangleright, A^+)$ be an affinoid ring, and let $M_1,\dots,M_n$ be finite subsets of $A^\triangleright$ such that $M_i \cdot A^\triangleright$ is open in $A^\triangleright$ for $i = 1,\dots,n$. Then $B := \{\sum a_\nu X^\nu \in A^\triangleright\langle X\rangle_M \mid a_\nu \in M_1^{\nu_1} \cdot \ldots \cdot M_n^{\nu_n} \cdot (A^+)^\wedge \text{ for every } \nu = (\nu_1,\dots,\nu_n) \in \mathbb{N}_0^n\}$ is a subring of $A^\triangleright\langle X\rangle_M$. The integral closure C of B in $A^\triangleright\langle X\rangle_M$ is a ring of integral elements of $A^\triangleright\langle X\rangle_M$. The affinoid ring $(A^\triangleright\langle X\rangle_M, C)$ is denoted by $A\langle X\rangle_M = A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n}$. We put $A\langle X_1,\dots,X_n\rangle := A\langle X_1,\dots,X_n\rangle_{\{1\},\dots,\{1\}}$.

A ring homomorphism $f: B \to C$ between affinoid rings is called a quotient mapping if $f: B^\triangleright \to C^\triangleright$ is surjective, continuous and open and C^+ is the integral closure of $f(B^+)$ in $C^\triangleright$. A ring homomorphism $f: A \to B$ from A to a complete affinoid ring B is called of **topologically finite type** if there exist a $n \in \mathbb{N}_0$, finite subsets $M_1,\dots,M_n$ of $A^\triangleright$ with $M_i \cdot A^\triangleright$ open in $A^\triangleright$ for $i = 1,\dots,n$ and a quotient mapping $g: A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n} \to B$ such that $f = g \circ h$, where $h: A \to A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n}$ is the natural homomorphism of affinoid rings. Then we have analogously to (3.3)

Lemma 3.5 (i) (Universal property of $A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n}$) *Let $h: A \to A\langle X\rangle_M$ be the natural ring homomorphism. Then $h(m)X_i \in (A\langle X\rangle_M)^+$ for every $i \in \{1,\dots,n\}$ and $m \in M_i$, and if $f: A \to B$ is a continuous ring homomorphism from A to a complete affinoid ring B and $b_1,\dots,b_n$ are elements of $B^\triangleright$ such that $f(m)b_i \in B^+$ for every $i \in \{1,\dots,n\}$ and $m \in M_i$, then there exists a unique continuous ring homomorphism $g: A\langle X\rangle_M \to B$ with $f = g \circ h$ and $g(X_i) = b_i$ for $i = 1,\dots,n$.*
(ii) *Let $f: A \to B$ be a ring homomorphism from A to a complete affinoid ring B. Then the following conditions are equivalent*

(a) *f is of topologically finite type*

(b) *$f: A^\triangleright \to B^\triangleright$ is of topologically finite type and there exists an open subring C of B^+ such that B^+ is integral over C, $f(A^+) \subseteq C$ and $f: A^+ \to C$ is of topologically finite type.*

In particular, for elements $s_1, \ldots, s_n$ of $A^\triangleright$ and finite subsets $T_1, \ldots, T_n$ of $A^\triangleright$ with $T_i \cdot A^\triangleright$ open in $A^\triangleright$ for $i = 1, \ldots, n$, the natural ring homomorphism $A \to A\left\langle \frac{T_1}{s_1}, \ldots, \frac{T_n}{s_n} \right\rangle$ is of topologically finite type.

(iii) *If A is Tate then a ring homomorphism $f: A \to B$ from A to a complete affinoid ring B is of topologically finite type if and only if f factors through a quotient mapping $A\langle X_1, \ldots, X_n\rangle \to B$ for some $n \in \mathbb{N}_0$.*

(iv) *Let B, C be complete affinoid rings and $f: A \to B$ and $g: B \to C$ continuous ring homomorphisms. If f and g are of topologically finite type then $g \circ f$ is of topologically finite type, and if $g \circ f$ is of topologically finite type then g is of topologically finite type.*

Definition. *Let $f: X \to Y$ be a morphism between adic spaces. We say f is* **locally of finite type** *if for every $x \in X$ there exist an open affinoid neighbourhood U of x in X and an open affinoid subspace V of Y such that $f(U) \subseteq V$ and the morphism between affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ induced by f is of topologically finite type. If f is quasi-compact (i.e., for every quasi-compact open subset U of Y, $f^{-1}(U)$ is quasi-compact) and locally of finite type then f is called of* **finite type**.

If $f: X \to Y$ is locally of finite type then to every neighbourhood U' of a point $x \in X$ and every neighbourhood V' of $f(x)$ there exist open affinoid neighbourhoods U and V of x and $f(x)$ in U' and V' such that $f(U) \subseteq V$ and $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type (3.5.ii, iv). In [H2] we will show

Proposition 3.6 *Let X, Y be adic spaces and $f: X \to Y$ a morphism locally of finite type. Let U and V be open affinoid subspaces of X and Y with $f(U) \subseteq V$. We assume that $\mathcal{O}_Y(V)$ has a noetherian ring of definition or is a strongly noetherian Tate ring. Then the morphism of affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type.*

Now we consider fibre products. In (3.7) we consider fibre products in the category of adic spaces and in (3.8) we consider the "fibre product" $S \times_Y X$ of an adic space S and a scheme X both living over a scheme Y.

Proposition 3.7 *Let X, Y, S be adic spaces and $f: X \to S$ and $g: Y \to S$ morphisms. We assume that f is locally of finite type, g is adic, and every point $y \in Y$ has an open affinoid neighbourhood U in Y such that $\mathcal{O}_Y(U)$ has a noetherian ring of definition or is a strongly noetherian Tate ring. Then there exists in the category of adic spaces the fibre product $X \times_S Y$ of X and Y over S. The projection $X \times_S Y \to Y$ is locally of finite type and the projection $X \times_S Y \to X$ is adic.*

Proof. We can assume that $S = \mathrm{Spa}A, Y = \mathrm{Spa}B, X = \mathrm{Spa}C$ are affinoid, $B^\triangleright$ has a noetherian ring of definition or is a strongly noetherian Tate ring, and g and f are induced by ring homomorphisms $\lambda: A \to B$ and $\mu: A \to C$ such that λ is adic and μ is of topologically finite type. Then there exist finite subsets

$M_1,\dots,M_n$ of $A^\triangleright$ such that $M_i \cdot A^\triangleright$ is open in $A^\triangleright$ and μ factors through a quotient mapping $\sigma: A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n} \to C$. Since λ is adic, $\lambda(M_i)\cdot B^\triangleright$ is open in $B^\triangleright$ for $i = 1,\dots,n$. Let $\lambda': A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n} \to B\langle X_1,\dots,X_n\rangle_{\lambda(M_1),\dots,\lambda(M_n)} =: E$ be the natural extension of λ. We equip $D^\triangleright := E^\triangleright/\lambda'(\ker\sigma)\cdot E^\triangleright$ with the f-adic topology such that the canonical mapping $\pi: E^\triangleright \to D^\triangleright$ is continuous and open. Let D^+ be the integral closure of $\pi(E^+)$ in $D^\triangleright$. Then D^+ is a ring of integral elements of $D^\triangleright$. We have a commutative diagram of affinoid rings $(D := (D^\triangleright, D^+))$

$$(*) \qquad \begin{array}{ccc} D & \longleftarrow & C \\ \uparrow & & \uparrow \\ B & \longleftarrow & A \end{array}$$

which is cocartesian in the category of complete affinoid rings (3.5.i). By (3.4.ii, iii) $D^\triangleright$ has a noetherian ring of definition or is a strongly noetherian Tate ring. So by (2.2) we have an adic $\operatorname{Spa} D$ associated with D. The commutative diagram induced by $(*)$

$$\begin{array}{ccc} \operatorname{Spa} D & \longrightarrow & \operatorname{Spa} C \\ \downarrow & & \downarrow \\ \operatorname{Spa} B & \longrightarrow & \operatorname{Spa} A \end{array}$$

is cartesian in the category of adic spaces by (2.1.i). □

Proposition 3.8 *Let $_$ denote the forgetful functor from the category of adic spaces to the category of locally ringed spaces which assigns to an adic space $(X, \mathcal{O}_X, (v_x \mid x \in X))$ the locally ringed space $(X, \mathcal{O}_X)$.*

Let X, Y be schemes, S an adic space, and $f: X \to Y$ and $g: \underline{S} \to Y$ morphisms of locally ringed spaces. We assume that f is locally of finite type and that every $s \in S$ has an open affinoid neighbourhood U in S such that $\mathcal{O}_S(U)$ has a noetherian ring of definition or is a strongly noetherian Tate ring. Then there exist an adic space R, a morphism of adic spaces $p: R \to S$ and a morphism of locally ringed spaces $q: \underline{R} \to X$ such that the diagram

$$\begin{array}{ccc} \underline{R} & \xrightarrow{q} & X \\ {\scriptstyle \underline{p}}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{S} & \xrightarrow[g]{} & Y \end{array}$$

commutes and the following universal property is satisfied. If U is an adic space, $u: U \to S$ a morphism of adic spaces and $v: \underline{U} \to X$ a morphism of locally ringed spaces with $g \circ \underline{u} = f \circ v$ then there exists a unique morphism of adic spaces $w: U \to R$ with $u = p \circ w$ and $v = q \circ \underline{w}$.

The morphism $p: R \to S$ is locally of finite type. We call R the fibre product of X and S over Y and denote it by $X \times_Y S$. Every morphism $f: X_1 \to X_2$ between schemes locally of finite type over Y induces by the universal property a morphism $X_1 \times_Y S \to X_2 \times_Y S$ of adic spaces over S which we denote by $f_{(S)}$.

Proof. We may assume that $Y = \operatorname{Spec} B, X = \operatorname{Spec} B[X_1,\dots,X_n]/I, S = \operatorname{Spa} A, A^\triangleright$ has a noetherian ring of definition or is a strongly noetherian Tate ring, and g is induced by a ring homomorphism $\lambda: B \to A^\triangleright$. Let E be a finite set of topologically nilpotent elements of $A^\triangleright$ such that $E \cdot A^\triangleright$ is open. For every

$k \in \mathbb{N}$ let $A(k) := A\langle X_1, \ldots, X_n\rangle_{E(k),\ldots,E(k)}$ with $E(k) := \{e_1 \cdot \ldots \cdot e_k \mid e_i \in E$ for $i = 1, \ldots, k\}$, and let $\lambda_k : B[X_1, \ldots, X_n] \to A(k)^\triangleright$ be the extension of λ with $\lambda_k(X_i) = X_i$ for $i = 1, \ldots, n$. Let $\pi_k : A(k)^\triangleright \to A(k)^\triangleright / \lambda_k(I) \cdot A(k)^\triangleright =: A_k^\triangleright$ be the natural mapping. We equip $A_k^\triangleright$ with the f-adic topology such that π_k is continuous and open. Then $A_k^\triangleright$ has a noetherian ring of definition or is a strongly noetherian Tate ring (3.4.ii, iii). Let A_k^+ be the integral closure of $\pi_k(A(k)^+)$ in $A_k^\triangleright$. Then $A_k := (A_k^\triangleright, A_k^+)$ is an affinoid ring. By (2.2) we have an adic space R_k associated with A_k. For $k \leqq h$, let $\varphi_{kh} : R_k \to R_h$ be the morphism which is induced by the continuous A-homomorphism $A_h \to A_k$ with $\pi_h(X_i) \mapsto \pi_k(X_i)$ for $i = 1, \ldots, n$ (3.5.i). Then φ_{kh} is an isomorphism of R_k onto the rational subset $\{x \in R_h \mid v_x(e\pi_h(X_i)) \leqq 1$ for every $e \in E(k), i = 1, \ldots, n\}$ of R_h. Hence there exists in the category of adic spaces over S the inductive limit R of the system $(R_k, \varphi_{kh} \mid k, h \in \mathbb{N})$. Let $p : R \to S$ be the structure morphism. The ring homomorphisms $\pi_k \circ \lambda_k : B[X_1, \ldots, X_n] \to A_k^\triangleright$ induce morphisms of locally ringed spaces $(R_k)_- \to X$ which glue together to a morphism of locally ringed spaces $q : \underline{R} \to X$. We have $g \circ \underline{p} = f \circ q$. Using (2.1.i) and (3.5.i), one can easily check that R, p, q satisfy the universal property. □

Lemma 3.9 (i) *Let*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & S \end{array}$$

be a cartesian square as in (3.7). *Then, for every $x \in X$ and $y \in Y$ with $f(x) = g(y)$, there exists a $z \in X \times_S Y$ with $x = q(z)$ and $y = p(z)$.*

(ii) *Let*

$$\begin{array}{ccc} \underline{R} & \xrightarrow{q} & X \\ {\scriptstyle \underline{p}}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{S} & \xrightarrow[g]{} & Y \end{array}$$

be the commutative diagram of (3.8). *Then, for every $x \in X$ and $s \in S$ with $f(x) = g(s)$, there exists a $r \in R$ with $x = q(r)$ and $s = p(r)$.*

Proof. (i) We use the notations of the proof of (3.7). Let A_0, B_0, C_0 be rings of definition of $A^\triangleright$, $B^\triangleright$, $C^\triangleright$ with $\lambda(A_0) \subseteq B_0$ and $\mu(A_0) \subseteq C_0$, and let I be an ideal of definition of A_0. Let F be the image of $B_0 \otimes_{A_0} C_0$ in $E^\triangleright := B^\triangleright \otimes_{A^\triangleright} C^\triangleright$, and let E^+ be the integral closure of the image of $B^+ \otimes_{A^+} C^+$ in $E^\triangleright$. We equip $E^\triangleright$ with the group topology such that $\{I^n \cdot F \mid n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0. Then $E^\triangleright$ is a f-adic ring and E^+ is a ring of integral elements of $E^\triangleright$. The completion of the affinoid ring $E = (E^\triangleright, E^+)$ is denoted by $B \hat{\otimes}_A C$. The commutative diagram

$$\begin{array}{ccc} B\hat{\otimes}_A C & \longleftarrow & C \\ \uparrow & & \uparrow \\ B & \longleftarrow & A \end{array}$$

is cocartesian in the category of complete affinoid rings. Hence $X \times_S Y = \mathrm{Spa} B \hat{\otimes}_A C = \mathrm{Spa} E$.

Let x be a point of X and y a point of Y with $f(x) = g(y)$. We consider x and y as valuations of $C^\triangleright$ and $B^\triangleright$. There exists a valuation v of $E^\triangleright$ which

lies over x and y. Let $\langle \Gamma_x \rangle$ and $\langle \Gamma_y \rangle$ be the convex hulls of the value groups Γ_x and Γ_x of x and y in the value group Γ_v of v. Put $H := \langle \Gamma_x \rangle \cup \langle \Gamma_y \rangle$ (hence $H = \langle \Gamma_x \rangle$ or $H = \langle \Gamma_y \rangle$). Then H contains the characteristic subgroup $c\Gamma_v$ of v (cf. [H1, Sect. 2]). Therefore we have the valuation $w := v|H$ of $E^\flat$ (cf. [H1, Sect. 2]). Also w lies over x and y. We have $w(e) \leqq 1$ for every $e \in E^+$, and $w(i)$ is cofinal in Γ_w for every $i \in I$ (since $x(i)$ is cofinal in Γ_x and $y(i)$ is cofinal in Γ_y). Hence $w \in \mathrm{Spa}E$.

(ii) We may assume $Y = \mathrm{Spec}B$, $X = \mathrm{Spec}C$, $S = \mathrm{Spa}A$, and f and g are given by ring homomorphisms $B \to C$ and $B \to A^\flat$. Let $\mathfrak{p} \in \mathrm{Spec}C$ be a prime ideal of C and $v \in \mathrm{Spa}A$ a valuation of $A^\flat$ with $\mathfrak{p} \cap B = \mathrm{supp}(v) \cap B$. We show that there exists a $r \in R$ with $\mathfrak{p} = q(r)$ and $v = p(r)$. We choose a prime ideal $\mathfrak{q}$ of $C \otimes_B A^\flat$ with $\mathfrak{p} = \mathfrak{q} \cap C$ and $\mathrm{supp}(v) = \mathfrak{q} \cap A^\flat$, and a valuation r of $C \otimes_B A^\flat$ such that $\mathfrak{q} = \mathrm{supp}(r)$, r extends the valuation v of $A^\flat$ and the value groups Γ_v and Γ_r of v and r have the same divisible hull (in particular, Γ_r is the convex hull of Γ_v in Γ_r). Let A_0 be a ring of definition of $A^\flat$, I a finitely generated ideal of definition of A_0, and $C' \subseteq C$ a finite set of generators of C over B. We choose a $k \in \mathbb{N}$ such that $r(c \otimes i) \leqq 1$ for every $c \in C'$ and $i \in I^k$. Then let D_0 denote the subring $A_0[c \otimes i | c \in C', i \in I^k]$ of $D^\flat := C \otimes_B A^\flat$, and let D^+ denote the integral closure of $A^+[c \otimes i | c \in C', i \in I^k]$ in $D^\flat$. We equip $D^\flat$ with the group topology such that $\{I^n \cdot D_0 | n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0. Then $D^\flat$ is a f-adic ring and D^+ is a ring of integral elements of $D^\flat$. We have $r \in \mathrm{Spa}(D^\flat, D^+)$, and $\mathrm{Spa}(D^\flat, D^+)$ is an open subspace of R. Hence r is an element of R with $\mathfrak{p} = q(r)$ and $v = p(r)$. □

4 Formal schemes, rigid analytic varieties and adic spaces

In this paragraph we construct a functor $t: \mathscr{F} \to \mathscr{A}$ form the category $\mathscr{F}$ of locally noetherian formal schemes to the category $\mathscr{A}$ of adic spaces, and a functor $r_k: \mathscr{R}_k \to \mathscr{A}$ from the category of rigid analytic varieties over a complete non-archimedean valued field k to the category of adic spaces. The idea of the definition of t and r_k is the following. First we notice that if A is a noetherian adic ring or a Tate ring of topologically finite type over k then, for every ring A^+ of integral elements of A, we have an adic space $\mathrm{Spa}(A, A^+)$ associated with the affinoid ring (A, A^+) (2.2). So we can define t and r_k on the affine objects of the categories $\mathscr{F}$ and $\mathscr{R}_k$, namely if $X = \mathrm{Spf}A$ then $t(X) := \mathrm{Spa}(A, A)$ and if $X = \mathrm{Sp}A$ then $r_k(X) := \mathrm{Spa}(A, A^\circ)$. We will see that every open subset U of an affine object X of $\mathscr{F}$ (resp. $\mathscr{R}_k$) induces in a natural way an open subset $a(U)$ of $t(X)$ (resp. $r_k(X)$), and every morphism $f: U_1 \to U_2$ between two open subspaces of affine objects X_1, X_2 of $\mathscr{F}$ (resp. $\mathscr{R}_k$) induces in a natural way a morphism of adic spaces $a(f): a(U_1) \to a(U_2)$. Then it is obvious how one has to define t and r_k on general objects, namely if an object X of $\mathscr{F}$ (resp. $\mathscr{R}_k$) is obtained by glueing affine objects $(X_i | i \in I)$ along isomorphisms $\varphi_{ji}: U_{ij} \to U_{ji}$ then $t(X)$ (resp. $r_k(X)$) is defined to be the adic space obtained by glueing $(t(X_i) | i \in I)$ (resp. $(r_k(X_i) | i \in I)$) along the isomorphisms $a(\varphi_{ji}): a(U_{ij}) \to a(U_{ji})$.

In the following proposition we construct the functor t. If X is an adic space then we equip, for every open subset U of X, $\mathcal{O}_X^+(U)$ with the subspace topology of $\mathcal{O}_X(U)$. Then $(X, \mathcal{O}_X^+)$ is a locally and topologically ringed space.

Proposition 4.1 *Let X be a locally noetherian formal scheme. Then there exist an adic space $t(X)$ and a morphism of locally and topologically ringed spaces $\pi = \pi_X : (t(X), \mathcal{O}_{t(X)}^+) \to (X, \mathcal{O}_X)$ such that the following universal property is satisfied. If Z is an adic space and $\lambda : (Z, \mathcal{O}_Z^+) \to (X, \mathcal{O}_X)$ is a morphism of locally and topologically ringed spaces then there exists a unique morphism between adic space $f : Z \to t(X)$ such that the diagram*

$$\begin{array}{ccc} (Z, \mathcal{O}_Z^+) & & \\ {\scriptstyle\mu}\downarrow & \searrow^{\lambda} & \\ (t(X), \mathcal{O}_{t(X)}^+) & \xrightarrow[\pi]{} & (X, \mathcal{O}_X) \end{array}$$

commutes, where μ is the morphism induced by f.

If $f : X \to Y$ is a morphism between locally noetherian formal schemes then by the universal property there exists a unique morphism between adic spaces $t(f) : t(X) \to t(Y)$ such that the diagram

$$\begin{array}{ccc} (t(X), \mathcal{O}_{t(X)}^+) & \xrightarrow{\pi_X} & (X, \mathcal{O}_X) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle f} \\ (t(Y), \mathcal{O}_{t(Y)}^+) & \xrightarrow[\pi_Y]{} & (Y, \mathcal{O}_Y) \end{array}$$

commutes, where μ is the morphism induced by $t(f)$. So we have a functor t from the category of locally noetherian formal schemes to the category of adic spaces.

Proof. We may assume that $X = \mathrm{Spf} A$ is affine. Then we put $t(X) := \mathrm{Spa}(A, A)$. By the following point (1), the identity $A \to A$ induces a morphism of locally and topologically ringed spaces $\pi : (t(X), \mathcal{O}_{t(X)}^+) \to (X, \mathcal{O}_X)$. By (2.1.ii) and (1), $t(X)$ and π satisfy the universal property.

(1) Let Y be an adic space. Then the morphisms of locally and topologically ringed spaces $(Y, \mathcal{O}_Y^+) \to (X, \mathcal{O}_X)$ correspond bijectively to the continuous ring homomorphisms $\mathcal{O}_X(X) \to \mathcal{O}_Y^+(Y)$.

Proof. We may assume that $Y = \mathrm{Spa}(B, B^+)$ is the adic space associated with a complete affinoid ring (B, B^+). Let a continuous ring homomorphism $\varphi : \mathcal{O}_X(X) \to \mathcal{O}_Y^+(Y)$ be given. We will construct a morphism of locally and topologically ringed spaces $f : (Y, \mathcal{O}_Y^+) \to (X, \mathcal{O}_X)$ such that $f^* : \mathcal{O}_X(X) \to \mathcal{O}_Y^+(Y)$ is equal to φ. For every $y \in Y$ put $g(y) = \{a \in \mathcal{O}_X(X) \mid v_y(\varphi(a)) < 1\}$. Then $g(y)$ is an open prime ideal of $\mathcal{O}_X(X) = A$. Hence we have a mapping $g : Y \to X$. This mapping is continuous, since, for every $s \in \mathcal{O}_X(X)$, $g^{-1}(\{x \in X \mid s(x) \neq 0\}) = \{y \in Y \mid 1 \leqq v_y(\varphi(a))\}$.

Let $s \in \mathcal{O}_X(X)$ be given. Put $U = \{x \in X | s(x) \neq 0\}$. Then $\mathcal{O}_X(U) = A\langle\frac{1}{s}\rangle$ and $\mathcal{O}_Y^+(g^{-1}(U)) = B^+\langle\frac{1}{\varphi(s)}\rangle$. Hence by (1.2.i) there exists a unique continuous ring homomorphism $\varphi_U: \mathcal{O}_X(U) \to \mathcal{O}_Y^+(g^{-1}(U))$ such that the diagram

$$\begin{array}{ccc} \mathcal{O}_Y^+(g^{-1}(U)) & \xleftarrow{\varphi_U} & \mathcal{O}_X(U) \\ \uparrow & & \uparrow \\ \mathcal{O}_Y^+(Y) & \xleftarrow[\varphi]{} & \mathcal{O}_X(X) \end{array}$$

commutes. The φ_U define a morphism of sheaves $\psi: \mathcal{O}_X \to g_*\mathcal{O}_Y^+$. It remains to show that, for every $y \in Y$, the ring homomorphism $\psi_y: \mathcal{O}_{X,g(y)} \to \mathcal{O}_{Y,y}^+$ induced by ψ is local. Let $\mathfrak{m}_y$ and $\mathfrak{m}_{g(y)}$ be the maximal ideals of $\mathcal{O}_{Y,y}^+$ and $\mathcal{O}_{X,g(y)}$, and let $h: \mathcal{O}_X(X) \to \mathcal{O}_{X,g(y)}$ be the natural ring homomorphism. Then by definition of g we have $h^{-1}(\mathfrak{m}_{g(y)}) = h^{-1}(\psi_y^{-1}(\mathfrak{m}_y))$ which implies $\mathfrak{m}_{g(y)} = \psi_y^{-1}(\mathfrak{m}_y)$. □

We call a subcategory $\mathscr{C}$ of the category $\mathscr{A}$ of adic spaces saturated if it satisfies the following properties. If $X \in \mathscr{C}$ and $U \subseteq X$ is an open subspace then $U \in \mathscr{C}$; if $X \in \mathscr{A}$ has an open covering $(U_i)_{i \in I}$ with $U_i \in \mathscr{C}$ for every $i \in I$ then $X \in \mathscr{C}$; if $X \in \mathscr{A}$ is isomorphic to a $Y \in \mathscr{C}$ then $X \in \mathscr{C}$; $\mathscr{C}$ is a full subcategory of $\mathscr{A}$.

Proposition 4.2 (i) *The functor t: $\mathscr{F} \to \mathscr{A}$ from the category $\mathscr{F}$ of locally noetherian formal schemes to the category of adic spaces is fully faithful. A morphism $f: X \to Y$ in $\mathscr{F}$ is adic* [EGA*, I.10.12] *resp. locally of finite type* [EGA*, I.10.13] *if and only if $t(f): t(X) \to t(Y)$ is adic resp. locally of finite type.*

(ii) *Let $\mathscr{C}$ be the smallest subcategory of $\mathscr{A}$ such that $t(\mathscr{F}) \subseteq \mathscr{C}$ and $\mathscr{C}$ is saturated. Then the objects of $\mathscr{C}$ are the adic spaces X with the property that every $x \in X$ has an open affinoid neighbourhood U such that $\mathcal{O}_X(U)$ has a noetherian ring of definition A such that $\mathcal{O}_X(U)$ is finitely generated over A and $\mathcal{O}_X^+(U)$ is the integral closure of A in $\mathcal{O}_X(U)$.*

Proof. (i) For every locally noetherian formal scheme Z we have the locally ringed space $Z_0 := (Z_t, \mathcal{O}_{t(Z)}^+|Z_t)$ with $Z_t = \{z \in t(Z) | v_z \text{ is a trivial valuation}\} \subseteq t(Z)$. The morphism of locally ringed spaces $j_Z: Z_0 \to Z$ induced by π_Z is an isomorphism. Let X, Y be locally noetherian formal schemes. Let $f: t(X) \to t(Y)$ be a morphism of adic spaces. Then $f(X_t) \subseteq Y_t$, and hence f induces a morphism of locally ringed spaces $f_0: X_0 \to Y_0$. Let $g: X \to Y$ be the morphism of locally ringed spaces with $g \circ j_X = j_Y \circ f_0$. Then g is a morphism of formal schemes (since X is locally noetherian), and it is easily seen that $f = t(g)$. Let $g_1, g_2: X \to Y$ be morphisms of formal schemes with $t(g_1) = t(g_2) =: f$. We have $g_1 \circ j_X = j_Y \circ f_0$ and $g_2 \circ j_X = j_Y \circ f_0$ which implies $g_1 = g_2$. Hence the functor t is fully faithful.

Let $f: X \to Y$ be a morphism between locally noetherian formal schemes. (3.3.ii) shows that a ring homomorphism $g: A \to B$ between noetherian complete adic rings is of topologically finite type (in the sense as defined in Sect. 3) if and only if g factors through a continuous and open mapping $A\langle X_1, \ldots, X_n\rangle \to B$. Then (3.2.i) and (3.6) imply that f is adic (resp. locally of finite type) if and only if $t(f)$ is adic (resp. locally of finite type).

(A more detailed proof would show that one can avoid the use of (3.2.i) and (3.6)).

(ii) Let $\mathscr{D}$ be the full subcategory of $\mathscr{A}$ consisting of the objects with the property described in (ii). Then $t(\mathscr{F}) \subseteq \mathscr{D}$ and $\mathscr{D}$ is saturated. Let $\mathscr{E}$ be a subcategory of $\mathscr{A}$ such that $t(\mathscr{F}) \subseteq \mathscr{E}$ and $\mathscr{E}$ is saturated. We have to show $\mathscr{D} \subseteq \mathscr{E}$. Let A be an affinoid ring such that there exist a noetherian ring of definition B of $A^\triangleright$ and a finite subset $E = \{e_1, \ldots, e_n\}$ of $A^\triangleright$ such that $A^\triangleright = B[E]$ and A^+ is the integral closure of B in $A^\triangleright$. We show that every point x of Spa$A$ has an open neighbourhood U in Spa$A$ with $U \in \mathscr{E}$. We assume that $v_x(e_i) > 1$ for $i = 1, \ldots, s$ and $v_x(e_i) \leqq 1$ for $i = s+1, \ldots, n$. We choose $b_1, \ldots, b_r \in B$ such that $\{b_1, \ldots, b_r\} \cdot B$ is an ideal of definition of B and $v_x(b_i e_j) \leqq 1$ for all $i = 1, \ldots, r, j = 1, \ldots, s$. Let C be the subring $B[e_{s+1}, \ldots, e_n]\left[\frac{1}{e_1}, \ldots, \frac{1}{e_s}\right]$ of the localization $A^\triangleright_{e_1, \ldots, e_s}$. We equip C with the adic topology such that $\{b_1, \ldots, b_r\} \cdot C$ is an ideal of definition. Let U be the rational subset $\{y \in \text{Spa}A \mid 1 \leqq v_y(e_i)$ for $i = 1, \ldots, s$ and $v_y(e_i) \leqq 1$ for $i = s+1, \ldots, n$ and $v_y(b_i e_j) \leqq 1$ for $i = 1, \ldots, r, j = 1, \ldots, s\}$ of Spa$A$. Then $x \in U$ and U is isomorphic to the rational subspace $\{y \in \text{Spa}(C, C) \mid v_y(b_i) \leqq v_y\left(\frac{1}{e_j}\right) \neq 0$ for $i = 1, \ldots, r, j = 1, \ldots, s\}$ of Spa(C, C). □

Let k be a field equipped with the complete topology of a rank 1 valuation of k. Let $\mathscr{R}_k$ be the category of rigid analytic varieties over k as defined in [BGR, 9.3.1]. Let $\mathscr{A}_k$ be the category of adic spaces over Spa(k, k°). (Every object of $\mathscr{A}_k$ is analytic in the sense of Sect. 3). We will construct a natural functor $r_k: \mathscr{R}_k \to \mathscr{A}_k$. For every $X \in \mathscr{R}_k$ let $|X|$ be the Grothendieck topology of X. Then $(|X|, \mathcal{O}_X)$ is a ringed site [SGA, IV.13.1], and every morphism $f: X \to Y$ in $\mathscr{R}_k$ induces a morphism of ringed sites $\tilde{f}: (|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$ [SGA, IV.13.3]. Similarly, for every $X \in \mathscr{A}_k$ let $|X|$ be the topology underlying X. Then $(|X|, \mathcal{O}_X)$ is a ringed site, and every morphism $f: X \to Y$ in $\mathscr{A}_k$ induces a morphism of ringed sites $\tilde{f}: (|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$. If $X = \text{Spa}(k, k^\circ)$ and $Y = \text{Sp}k$ then $(|X|, \mathcal{O}_X) = (|Y|, \mathcal{O}_Y)$, and we denote this ringed site by S. If X is an object of $\mathscr{R}_k$ or $\mathscr{A}_k$ then $(|X|, \mathcal{O}_X)$ is a ringed site over S, and if f is a morphism in $\mathscr{R}_k$ or $\mathscr{A}_k$ then $\tilde{f}$ is a morphism of ringed sites over S, i.e, we have functors from the categories $\mathscr{R}_k$ and $\mathscr{A}_k$ to the category of ringed sites over S. In the following proposition we construct the functor $r_k: \mathscr{R}_k \to \mathscr{A}_k$.

Proposition 4.3 *To every $X \in \mathscr{R}_k$ there exist a $r_k(X) \in \mathscr{A}_k$ and a morphism $\rho = \rho_X: (|r_k(X)|, \mathcal{O}_{r_k(X)}) \to (|X|, \mathcal{O}_X)$ of ringed sites over S such that $r_k(X)$ is locally of finite type over* Spa(k, k°), *ρ is locally coherent* [SGA, VI.3.7] *and the following universal property is satisfied. If Z is an adic space locally of finite type over* Spa(k, k°) *and $\lambda: (|Z|, \mathcal{O}_Z) \to (|X|, \mathcal{O}_X)$ is a locally coherent morphism between ringed sites over S then there exists a unique morphism $f: Z \to r_k(X)$ in $\mathscr{A}_k$ such that the diagram of ringed sites*

$$\begin{array}{ccc} (|Z|, \mathcal{O}_Z) & & \\ \downarrow \tilde{f} & \searrow^{\lambda} & \\ (|r_k(X)|, \mathcal{O}_{r_k(X)}) & \xrightarrow[\rho]{} & (|X|, \mathcal{O}_X) \end{array}$$

commutes.

If $f: X \to Y$ is a morphism in $\mathscr{R}_k$ then $\bar{f}: (|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$ is locally coherent, and hence by the universal property there exists a unique morphism $r_k(f): r_k(X) \to r_k(Y)$ in $\mathscr{A}_k$ such that the diagram

$$\begin{array}{ccc} (|r_k(X)|, \mathcal{O}_{r_k(X)}) & \xrightarrow{\rho_X} & (|X|, \mathcal{O}_X) \\ \overline{r_k(f)} \downarrow & & \downarrow \bar{f} \\ (|r_k(Y)|, \mathcal{O}_{r_k(Y)}) & \xrightarrow[\rho_Y]{} & (|Y|, \mathcal{O}_Y) \end{array}$$

commutes. Thus we have a functor $r_k: \mathscr{R}_k \to \mathscr{A}_k$.

Proof. We may assume that $X = \mathrm{Sp}A$ is the affinoid rigid analytic variety associated with a Tate ring A of topologically finite type over k. We put $r_k(X) := \mathrm{Spa}(A, A^\circ)$. By the subsequent lemma $r_k(X)$ is of finite type over $\mathrm{Spa}(k, k^\circ)$. By [H1, 4.3] there is a natural one-to-one correspondence between the rational subsets of $\mathrm{Sp}A$ and the rational subsets of $\mathrm{Spa}(A, A^\circ)$. This correspondence extends in a unique way to a morphism of sites $m: |r_k(X)| \to |X|$. m is coherent. For every rational subset U of X and corresponding rational subset V of $r_k(X)$ there exists a unique A-algebra homomorphism $\Theta_U: \mathcal{O}_X(U) \to \mathcal{O}_{r_k(X)}(V)$, and Θ_U is bijective. The Θ_U's induce an isomorphism of sheaves $\Theta: \mathcal{O}_X \to m_* \mathcal{O}_{r_k(X)}$. Then $\rho = (m, \Theta): (|r_k(X)|, \mathcal{O}_{r_k(X)}) \to (|X|, \mathcal{O}_X)$ is a morphism of ringed sites over S. We check the universal property. Let Z be an adic space locally of finite type over $\mathrm{Spa}(k, k^\circ)$ and $\lambda: (|Z|, \mathcal{O}_Z) \to (|X|, \mathcal{O}_X)$ a locally coherent morphism between ringed sites over S. There exists a unique morphism of ringed sites $\mu: (|Z|, \mathcal{O}_Z) \to (|r_k(X)|, \mathcal{O}_{r_k(X)})$ with $\lambda = \rho \circ \mu$. Let $g: (Z, \mathcal{O}_Z) \to (r_k(X), \mathcal{O}_{r_k(X)})$ be the morphism of ringed spaces whose associated morphism of ringed sites is μ (cf. [SGA, IV.4.2.3]). We have to show that g is a morphism in $\mathscr{A}_k$. We know that g is a morphism of ringed spaces over k and that for every open quasi-compact subset U of Z the restriction $g: U \to r_k(X)$ is a quasi-compact mapping. Since the k-algebra homomorphisms between Tate rings of topologically finite type over k are continuous [BGR, 6.1.3], we see that $\mathcal{O}_{r_k(X)} \to g_* \mathcal{O}_Z$ is a morphism of topological sheaves. It remains to show that, for every $z \in Z$, the ring homomorphism $g_z^*: \mathcal{O}_{r_k(X), g(z)} \to \mathcal{O}_{Z,z}$ is compatible with the valuations $v_{g(z)}$ and v_z. For that we note the following property

(1) Let $U \subseteq r_k(X)$ be open, $f \in \mathcal{O}_{r_k(X)}(U)$, $a \in k^*$. We consider the sets $U_1 = \{x \in U | v_x(f) < v_x(a)\}, U_2 = \{x \in U | v_x(f) \leqq v_x(a)\}, V_1 = \{x \in g^{-1}(U) | v_x(g^*(f)) < v_x(a)\}, V_2 = \{x \in g^{-1}(U) | v_x(g^*(f)) \leqq v_x(a)\}$. Then $V_1 = g^{-1}(U_1)$ and $V_2 = g^{-1}(U_2)$.

Proof. Let M be the set of points $z \in Z$ such that the residue class field of $\mathcal{O}_{Z,z}$ is finite over k. For every $z \in M$ the ring homomorphism $g_z^*: \mathcal{O}_{r_k(X), g(z)} \to \mathcal{O}_{Z,z}$ is compatible with the valuations $v_{g(z)}$ and v_z, since g_z^* is local and the valuation ring k° of k extends uniquely to every finite extension field of k. Hence $V_1 \cap M = g^{-1}(U_1) \cap M$ and $V_2 \cap M = g^{-1}(U_2) \cap M$. Then [H1, 4.3] and the subsequent lemma imply $V_1 = g^{-1}(U_1)$ and $V_2 = g^{-1}(U_2)$.

Let $z \in Z$ be given. We consider the ring homomorphism $g_z^*: \mathcal{O}_{r_k(X), g(z)} \to \mathcal{O}_{Z,z}$. Let $\mathfrak{m}_{g(z)}$ and $\mathfrak{m}_z$ be the maximal ideals of $\mathcal{O}_{r_k(X), g(z)}$ and $\mathcal{O}_{Z,z}$. Then by (1), $g_z^*(\mathfrak{m}_{g(z)}) = g_z^*(\bigcap_{a \in k^*} \{f \in \mathcal{O}_{r_k(X), g(z)} | v_{g(z)}(f) \leqq v_{g(z)}(a)\}) \subseteq \bigcap_{a \in k^*} \{f \in \mathcal{O}_{Z,z} | v_z(f) \leqq v_z(a)\} = \mathfrak{m}_z$, i.e., g_z is a local ring homomorphism.

Furthermore, (1) implies that if f is an element of $\mathcal{O}_{r_k(X),g(z)}$ with $v_{g(z)}(f) \leqq 1$ (resp. $v_{g(z)}(f) < 1$) then $v_z(g_z^*(f)) \leqq 1$ (resp. $v_z(g_z^*(f)) < 1$). This shows that g_z^* is compatible with the valuations $v_{g(z)}$ and v_z. □

Lemma 4.4 *Let A be a Tate ring of topologically finite type over k. Then there exists a unique ring A^+ of integral elements of A such that $k^\circ \subseteq A^+$ and the ring homomorphism of affinoid rings $(k, k^\circ) \to (A, A^+)$ is of topologically finite type, namely $A^+ = A^\circ$.*

Proof. Let A^+ be a ring of integral elements of A such that $f:(k, k^\circ) \to (A, A^+)$ is of topologically finite type. By (3.5.iii) f factors through a quotient mapping $g:(k, k^\circ)\langle X_1, \ldots, X_n\rangle \to (A, A^+)$. We have $(k, k^\circ)\langle X_1, \ldots, X_n\rangle^+ = k^\circ\langle X_1, \ldots, X_n\rangle = k\langle X_1, \ldots, X_n\rangle^\circ$. Since $g: k\langle X_1, \ldots, X_n\rangle \to A$ is surjective, A° is the integral closure of $g(k\langle X_1, \ldots, X_n\rangle^\circ)$ in A [BGR, 6.4.3]. Hence $A^+ = A^\circ$. This also shows that $(k, k^\circ) \to (A, A^\circ)$ is of topologically finite type. □

For an object X of $\mathscr{A}_k$ or $\mathscr{R}_k$ let $\mathrm{Shv}(X)$ denote the topos of the site $|X|$. Before (4.2) we defined the saturated subcategories of $\mathscr{A}$. Analogously we define the saturated subcategories of $\mathscr{A}_k$.

Proposition 4.5 (i) *For every $X \in \mathscr{R}_k$, the morphism of ringed toposes $(\mathrm{Shv}(r_k(X)), \mathcal{O}_{r_k(X)}) \to (\mathrm{Shv}(X), \mathcal{O}_X)$ induced by the morphism of ringed sites $\rho_X: (|r_k(X)|, \mathcal{O}_{r_k(X)}) \to (|X|, \mathcal{O}_k)$ is an equivalence of ringed toposes.*
(ii) *The functor $r_k: \mathscr{R}_k \to \mathscr{A}_k$ is fully faithful.*
(iii) *Let $\mathscr{C}$ be the smallest subcategory of $\mathscr{A}_k$ such that $r_k(\mathscr{R}_k) \subseteq \mathscr{C}$ and $\mathscr{C}$ is saturated. Then the objects of $\mathscr{C}$ are the adic spaces locally of finite type over* $\mathrm{Spa}(k, k^\circ)$.
(iv) *The functor $r_k: \mathscr{R}_k \to \mathscr{A}_k$ restricts to an equivalence between the category of quasi-separated rigid analytic varieties over k and the category of quasi-separated adic spaces locally of finite type over* $\mathrm{Spa}(k, k^\circ)$.
(v) *For every $X \in \mathscr{R}_k$, the adic space $r_k(X)$ is affinoid if and only if X is an affinoid rigid analytic variety over k.*

fully faithful.

Proof. (i)–(iv) follow from the proof of (4.3) (cf. [H1, 4.6] for (i)). Let $X \in \mathscr{R}_k$ such that $Y := r_k(X)$ is an affinoid adic space. By (3.6) the morphism of affinoid rings $(k, k^\circ) \to (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ is of topologically finite type. Then by (4.4) $\mathcal{O}_Y^+(Y) = \mathcal{O}(Y)^\circ$. The adic space $r_k(X)$ is isomorphic to $\mathrm{Spa}(\mathcal{O}_Y(Y), \mathcal{O}_Y(Y)^\circ) = r_k(\mathrm{Sp}\mathcal{O}_Y(Y))$ (over $\mathrm{Spa}(k, k^\circ)$). Then by (ii) X is isomorphic to $\mathrm{Sp}\mathcal{O}_Y(Y)$. □

Remark 4.6 (i) Let X be a scheme locally of finite type over k. One associates with X a rigid analytic variety X^{an} [BGR, 9.3.4]. The identity $k \to k$ induces a morphism of locally ringed spaces $\mathrm{Spa}(k, k^\circ)_- \to \mathrm{Spec}\, k$. Then by (3.8) we have the adic space $X \times_{\mathrm{Spec}\, k} \mathrm{Spa}(k, k^\circ)$ over $\mathrm{Spa}(k, k^\circ)$. It follows from our construction of the functor r_k and of $X \times_{\mathrm{Spec}\, k} \mathrm{Spa}(k, k^\circ)$ that

$$r_k(X^{\mathrm{an}}) \cong X \times_{\mathrm{Spec}\, k} \mathrm{Spa}(k, k^\circ),$$

where the isomorphism is an isomorphism over $\mathrm{Spa}(k, k^\circ)$.
(ii) Let V be a complete discrete valuation ring of rank 1, and let k be the quotient field of V. In [R] there is constructed a functor $s: \mathscr{D} \to \mathscr{R}_k$ from the category $\mathscr{D}$ of formal schemes locally of finite type over $\mathrm{Spf}\, V$ to the category

$\mathscr{R}_k$ of rigid analytic varieties over k. Let $\mathscr{E}$ be the category whose objects are the locally noetherian formal schemes and whose morphisms are the adic morphisms between formal schemes. By (3.2.ii) and (4.2.i) we have the functor $t_a: \mathscr{E} \to \mathscr{A}, X \to t(X)_a$ which assigns to every $X \in \mathscr{E}$ the open subspace $t(X)_a$ of analytic points of the adic space $t(X)$. Let $u: \mathscr{D} \to \mathscr{E}$ be the natural functor. Then the diagram

$$\begin{array}{ccc} \mathscr{D} & \xrightarrow{s} & \mathscr{R}_k \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle r_k} \\ \mathscr{E} & \xrightarrow[t_a]{} & \mathscr{A} \end{array}$$

commutes up to isomorphism, i.e., the functors $r_k \circ s$ and $t_a \circ u$ are isomorphic.
(iii) We use the notations of (ii). Let $f: X \to Y := \operatorname{Spf} V$ be a formal scheme locally of finite type over $\operatorname{Spf} V$. We consider the morphism of adic spaces $t(f): t(X) \to t(Y)$. The topological space Z underlying $t(Y)$ is the Sierpinski space, i.e., Z has an open generic point η (i.e., $\{\eta\}$ is open and $\overline{\{\eta\}} = Z$) and Z consists of two points. We have $\{\eta\} = t(Y)_a$, and the fibre $t(f)^{-1}(\eta)$ is the open subspace of analytic points of $t(X)$. Raynaud calls the rigid analytic variety $s(X)$ the generic fibre of f. So the diagram in (ii) says that the functor r_k transforms the generic fibre $s(X)$ of f into the generic fibre $t(f)^{-1}(\eta)$ of $t(f)$, i.e., $r_k(s(X)) \cong t(f)^{-1}(\eta)$.
(iv) If A is an adic noetherian ring then the identity $A \to A$ induces a morphism of locally ringed spaces $\operatorname{Spa}(A, A)_- \to \operatorname{Spec} A$. More general, if $\hat{X}$ is the formal completion of a locally noetherian scheme X along a closed subset $X' \subseteq X$ then there is a natural morphism of locally ringed spaces $\rho = \rho_{\hat{X}}: \underline{t(\hat{X})} \to X$. Namely, if $(X_i | i \in I)$ is an open affine covering of X and $\hat{X}_i$ denotes the formal completion of X_i along $X' \cap X_i$ then $(t(\hat{X}_i) | i \in I)$ is an open covering of $t(\hat{X})$ and the morphisms $\underline{t(\hat{X}_i)} \to X_i$ just described glue together to a morphism $\rho: \underline{t(\hat{X})} \to X$.

Let $f: X \to Y$ be a morphism between locally noetherian schemes, and let $X' \subseteq X$ and $Y' \subseteq Y$ be closed subsets with $f(X') \subseteq Y'$. Then f induces a morphism $\hat{f}: \hat{X} \to \hat{Y}$ between the formal completions of X and Y along X' and Y'. The diagram

$$(*) \qquad \begin{array}{ccc} \underline{t(\hat{X})} & \xrightarrow{\rho_{\hat{X}}} & X \\ {\scriptstyle \underline{t(\hat{f})}}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{t(\hat{Y})} & \xrightarrow[\rho_{\hat{Y}}]{} & Y \end{array}$$

is commutative. If f is locally of finite type then, by (3.8), $(*)$ induces a morphism of adic spaces

$$\varphi: t(\hat{X}) \to X \times_Y t(\hat{Y}) .$$

If $X' = f^{-1}(Y')$ then the following holds.

(a) $\operatorname{im}(\varphi) = \{x \in X \times_Y t(\hat{Y}) | x$ has a specialization y in $X \times_Y t(\hat{Y})$ such that v_y is a trivial valuation $\}$.
(b) φ is a local isomorphism.
(c) If f is separated then φ is an open embedding.
(d) If f is proper then φ is an isomorphism.

Lemma 1.3.16. *Let $f: X \to Y$ be a separated morphism of analytic adic spaces. Let x be a point of X such that $k(f(x))$ is dense in $k(x)$, and let $u, v \in X$ be specializations of x with $f(u) = f(v)$. Then $u = v$.*

Proof. Let A and B be the valuation rings of X such that x is the support of A and B, u is a center of A and v is a center of B ((1.3.6.i)). Since $f(u) = f(v)$, we have $f_v(A) = f_v(B)$. Since $k(f(x))$ is dense in $k(x)$, we obtain $A = B$. Hence $u = v$ ((1.3.7)). □

Remark 1.3.17. Let Y be an adic space such that every $y \in Y$ has an open affinoid neighbourhood U in Y such that the f-adic ring $\mathcal{O}_Y(U)$ has a noetherian ring of definition over which $\mathcal{O}_Y(U)$ is finitely generated, and let $f: X \to Y$ be a morphism of adic spaces which is of finite type and proper. Then, for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ on X, all direct image sheaves $R^n f_* \mathcal{F}$ are coherent $\mathcal{O}_Y$-modules ([Hu$_4$, 3.12.13]).

Remark 1.3.18. Let A be a complete valuation ring of rank 1, let $\mathcal{X}$ and $\mathcal{Y}$ be formal schemes which are flat and locally of finite type over A, and let $\mathfrak{f}: \mathcal{X} \to \mathcal{Y}$ be an A-morphism. Let $f: X \to Y$ be the associated morphism of rigid analytic varieties ([R$_1$]), and let $r(f): r(X) \to r(Y)$ be the morphism of adic spaces associated with f ((1.1.11)). Then ([Hu$_4$, 3.11, 3.13])

i) $r(f)$ is separated if and only if $\mathfrak{f}$ is separated.

ii) $r(f)$ is proper if and only if $\mathfrak{f}$ is proper.

iii) $r(f)$ is partially proper if and only if, for every irreducible closed subset Z of $\mathcal{X}$, the restriction $\mathfrak{f}: Z \to \mathcal{Y}$ is proper.

Remark 1.3.19. Let k be a complete non-archimedean field, and let $f: X \to Y$ be a morphism of rigid analytic varieties over k.
For affinoid open subspaces U, V of X and W of Y with $f(V) \subseteq W$, one puts $U \underset{W}{\subset\subset} V$ if there exists an affinoid generating system $f_1, \ldots, f_r$ of $\mathcal{O}_X(V)$ over $\mathcal{O}_Y(W)$ such that $U \subseteq \{x \in V \mid |f_i(x)| < 1 \text{ for } i = 1, \ldots, r\}$ ([BGR, 9.6.2]).

i) We call f *partially proper* if f is separated and if there exist an admissible open covering $(Y_i)_{i \in I}$ of Y by open affinoid subspaces Y_i, and, for every $i \in I$, an admissible open covering $(X_{ij})_{j \in J(i)}$ of $f^{-1}(Y_i)$ by open affinoid subspaces X_{ij} such that, for every $i \in I$ and $j \in J(i)$, there exists an open affinoid subspace X'_{ij} of $f^{-1}(Y_i)$ such that $X_{ij} \underset{Y_i}{\subset\subset} X'_{ij}$.
Then according to Kiehl's definition in [K], f is proper if and only if f is quasi-compact and partially proper.

ii) Let $r(f): r(X) \to r(Y)$ be the morphism of adic spaces associated with f. The following conditions are equivalent ([Hu$_4$, 3.11.21.ii])

Proof. Let $d: X' \to S$ and $e: Y \to S$ be the structure morphisms.
i) We fix a $x' \in X'$. Let $x \in X$ be a generalization of x' and let A be the valuation ring of X' with support x and center x' ((1.3.6.i)). Then $g(x')$ is a center of $f_*(A)$ on Y with $e(g(x')) = d(x')$. Since e is separated, we obtain that $g(x')$ is uniquely determined. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X,x} & \xleftarrow{\alpha} & \mathcal{O}_{X',x'} \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle g^*} \\ \mathcal{O}_{Y,f(x)} & \longleftarrow & \mathcal{O}_{Y,g(x')}. \end{array}$$

Since α is injective ((1.1.10.iii)), g^* is uniquely determined.
ii) Since X' is quasi-compact and Y quasi-separated, g is quasi-compact. Hence by the subsequent Lemma (1.3.15) it suffices to show that g is injective.
Let u', v' be points of X' with $g(u') = g(v')$. We choose $u, v \in X$ with $u \succ u'$, $v \succ v'$. Then $g(u) \succ g(u') = g(v')$ and $g(v) \succ g(v') = g(u')$. By (1.1.10.i) we may assume $g(u) \succ g(v)$, and then by (1.1.10.v) there exists a $v_0 \in X$ with $v_0 \succ v$ and $g(v_0) = g(u)$. Since $g|X = f$ is injective, we obtain $v_0 = u$. Thus we have $v_0 \succ v'$, $v_0 \succ u'$, $g(v') = g(u')$. The morphism g is separated, since X' is separated over S. Hence the subsequent Lemma (1.3.16) implies $u = v$.
iii) We choose affinoid open subsets L and M of Y and S such that $y \in L$, $e(L) \subseteq M$ and there exists a finite subset E of $\mathcal{O}_Y(L)$ such that $\mathcal{O}_Y^+(L)$ is the smallest ring of integral elements of $\mathcal{O}_Y(L)$ which contains E and the image of $\mathcal{O}_S^+(M)$ in $\mathcal{O}_Y(L)$. We consider the open subset $f^{-1}(L)$ of X. We have

(1) Let G denote the set of the generalizations of x' in X'. Then $G \cap X \subset f^{-1}(L)$.

Proof. Since $f(x) \succ y$, we have $x \in f^{-1}(L)$. By our assumption on x we have, for every $z \in G \cap X$, $z \succ x$ and hence $z \in f^{-1}(L)$.

Since the inclusion $X' \hookrightarrow X$ is quasi-compact, (1) implies that there exists an affinoid open neighbourhood V of x' in $d^{-1}(M)$ with $V \cap X \subseteq f^{-1}(L)$. Composing the mapping $f^*: \mathcal{O}_Y(L) \to \mathcal{O}_X(V \cap X)$ with the inverse mapping of $\mathcal{O}_{X'}(V) \xrightarrow{\sim} \mathcal{O}_X(V \cap X)$, we obtain a continuous ring homomorphism $\varphi: \mathcal{O}_Y(L) \to \mathcal{O}_{X'}(V)$. Let U be the rational subset $R(\frac{\varphi(E) \cup \{1\}}{1})$ of V, and let $\psi: \mathcal{O}_Y(L) \to \mathcal{O}_{X'}(U)$ be the composition of φ with the restriction mapping $\mathcal{O}_{X'}(V) \to \mathcal{O}_{X'}(U)$. Then $\psi(\mathcal{O}_Y^+(L)) \subseteq \mathcal{O}_{X'}^+(U)$, and hence ψ induces a morphism of adic spaces $g: U \to L$. Then $x' \in U$, $g|U \cap X = f|U \cap X$, $y = g(x')$, and g is a M-morphism. □

Lemma 1.3.15. *Let $f: X \to Y$ be an injective quasi-compact morphism between analytic adic spaces. Then f is a homeomorphism onto its image.*

Proof. $f(X)$ is a pro-constructible subset of Y. It suffices to show that the mapping $g: X \to f(X)$ is specializing. But this is obvious, since g is generalizing (by (1.1.10.v)) and injective. □

(a) $f: X \to Y$ is separated

(b) $r(f): r(X) \to r(Y)$ is separated.

iii) We consider the conditions

(a) $f: X \to Y$ is partially proper

(b) $r(f): r(X) \to r(Y)$ is partially proper.

Then (a) implies (b) and if the valuation ring k° is discrete then (b) implies (a). (The proof of the first statement is easy (see proof of (1.5.9)). The second statement follows from [L_1] and (1.3.18.iii).)

iv) With (iii) we obtain that if we consider the statements

(a) $f: X \to Y$ is proper

(b) $r(f): r(X) \to r(Y)$ is proper

then (a) implies (b) and if k° is discrete then (b) implies (a).

1.4 Finite morphisms

(1.4.1) If $A = (A^\circ, A^+)$ is an affinoid ring and J an ideal of A° then A/J denotes the affinoid ring $(A^\circ/J, (A^+/J \cap A^+)^c)$ where A°/J carries the quotient topology of A° and $(A^+/J \cap A^+)^c$ is the integral closure of $A^+/J \cap A^+$ in A°/J.

Let $X = (X, \mathcal{O}_X, (v_x \mid x \in X))$ be an adic space, and let $I \subseteq \mathcal{O}_X$ be a quasi-coherent $\mathcal{O}_X$-module. Put $Y := \{x \in X \mid I_x \neq \mathcal{O}_{X,x}\}$, $\mathcal{O}_Y := (\mathcal{O}_X/I) \mid Y$, and, for every $y \in Y$, let v'_y denote the valuation of $\mathcal{O}_{Y,y}$ induced by v_y. There is a unique ring topology on the sheaf $\mathcal{O}_Y$ such that, for every open affinoid subspace U of X, the mapping $\mathcal{O}_X(U) \to \mathcal{O}_Y(Y \cap U)$ is continuous and open. The triple $Y = (Y, \mathcal{O}_Y, (v'_y \mid y \in Y))$ is an adic space ([Hu$_4$, 3.6], [Hu$_2$]). If $X = \mathrm{Spa}A$ and $I = \tilde{I}(X, I)$ then $Y = \mathrm{Spa}A/I$. The adic spaces of the form $(Y, \mathcal{O}_Y, (v'_y \mid y \in Y))$ are called the *closed adic subspaces* of X.

A morphism of adic spaces $f: Z \to X$ is called a closed (resp. open, resp. locally closed) embedding if f has a factorization $f = i \circ g$ where $i: Y \to X$ is the natural morphism from a closed (resp. open, resp. locally closed) adic subspace Y of X to X and $g: Z \to Y$ is an isomorphism.

Let k be a complete non archimedean field. For every Tate ring A of topologically finite type over k and every ideal J of A, we have $(A, A^\circ)/J = (A/J, (A/J)^\circ)$ ([BGR, 6.3.4 Prop.1]). Therefore, a morphism $f: X \to Y$ of rigid analytic varieties over k is a closed embedding if and only if the associated morphism of adic spaces $r(f): r(X) \to r(Y)$ is a closed embedding.

(1.4.2) A ring homomorphism of complete affinoid rings $f: A \to B$ is called *finite* if f is of topologically finite type and the ring homomorphisms $A^\flat \to B^\flat$ and $A^+ \to B^+$ are integral. Then $B^\flat$ is finite over $A^\flat$, the topology of $B^\flat$ is the natural $A^\flat$-module topology ([Hu, §2]) and B^+ is the integral closure of A^+ in $B^\flat$. Conversely, if C is a finite $A^\flat$-algebra equipped with the natural $A^\flat$-module topology and if C^+ denotes the integral closure of A^+ in C then (C, C^+) is a complete affinoid ring which is finite over A ([Hu$_4$, 3.12.10]).

Lemma 1.4.3. *Let A and B be affinoid rings which are complete and Tate, and let $A \to B$ be a morphism of topologically finite type. Let $t_1, \ldots, t_n$ be elements of B^+ such that the A-morphism $p: A\langle T_1, \ldots, T_n\rangle \to B, T_i \mapsto t_i$ for $i = 1, \ldots, n$ is a quotient mapping* ([Hu, 3.5.iii]).
Equivalent are:

a) *B is finite over A.*

b) *For every $i \in \{1, \ldots, n\}$, there exists a monic polynomial $p_i \in A^+[X]$ with $p_i(t_i) \in (B^\flat)^{\circ\circ}$.*

Proof. If B is finite over A then B^+ is integral over A^+ and hence b) is satisfied. Now assume b), and we show that B is finite over A. Let A_0 be a ring of definition of A^+ which contains all coefficients of the p_i, and let I be an ideal of definition of A_0. The A_0-algebra $B_0 := p(A_0\langle T_1, \ldots, T_n\rangle)$ is a ring of definition of B^+ with $t_i \in B_0$ for $i = 1, \ldots, n$ and $A_0[t_1, \ldots, t_n]$ is dense in B_0. Since $p_i(t_i) \in (B^\flat)^{\circ\circ}$, there exists a $m \in \mathbb{N}$ with $p_i(t_i)^m \in IB_0$ for $i = 1, \ldots, n$. Replacing p_i by p_i^m, we may assume that $p_i(t_i) \in IB_0$ for $i = 1, \ldots, n$. Let $\bar{t}_i$ be the image of t_i in B_0/IB_0. Then $B_0/IB_0 = A_0[\bar{t}_1, \ldots, \bar{t}_n]$ and every $\bar{t}_i$ is integral over A_0. Hence B_0/IB_0 is finite over A_0. This implies that B_0 is finite over A_0 ([M, 28.P Lemma]). Hence B_0 is integral over A_0. Since $A\langle T_1, \ldots, T_n\rangle^+$ is the integral closure of $A^+ \cdot A_0\langle T_1, \ldots, T_n\rangle$ in $A\langle T_1, \ldots, T_n\rangle^\flat$, we obtain that B^+ is integral over A^+. Then also $B^\flat = A^\flat \otimes_{A^+} B^+$ is integral over $A^\flat$. □

(1.4.4) A morphism of adic spaces $f: X \to Y$ is called *finite* if, for every $y \in Y$, there exists an open affinoid neighbourhood V of y in Y such that $U := f^{-1}(V)$ is affinoid and the ring homomorphism of affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is finite. Then, for every open affinoid subspace V of Y, $U := f^{-1}(V)$ is affinoid and $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is finite over $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$ ([Hu$_4$, 3.6.20, 3.12.12], [Hu$_2$]).

Lemma 1.4.5. i) *Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

be a cartesian diagram of adic spaces. Then if f is finite then also f' is finite.
ii) *Every finite morphism of adic spaces is proper.*
iii) *Every closed embedding of adic spaces is finite.*
iv) *A morphism of rigid analytic varieties $f: X \to Y$ is finite if and only if the associated morphism of adic spaces $r(f): r(X) \to r(Y)$ is finite.*

Proof. i) We may assume that $X = \mathrm{Spa}A$, $Y = \mathrm{Spa}B$ and $Y' = \mathrm{Spa}B'$ are affinoid. Equip $A'^{\flat} := A^{\flat} \otimes_{B^{\flat}} B'^{\flat}$ with its natural $B'^{\flat}$-module topology ([Hu, §2]), and let A'^{+} denote the integral closure of B'^{+} in $A'^{\flat}$. Then $A' := (A'^{\flat}, A'^{+})$ is an affinoid ring which is finite over B', and $X \times_Y Y' = \mathrm{Spa}A'$.
ii) Let $f: X \to Y$ be a finite morphism. Obviously, f is separated. By i) it remains to show that f is closed. For that it suffices to show that f is specializing. But this follows from [HK, 2.1.2, 2.1.6]. (If Y is analytic then the assertion follows trivially from (1.3.9)).
iv) Let $f: \mathrm{Sp}A \to \mathrm{Sp}B$ be a morphism of affinoid rigid analytic varieties. Then $r(f): \mathrm{Spa}(A, A^{\circ}) \to \mathrm{Spa}(B, B^{\circ})$. If f is finite then A° is integral over B° ([BGR, 6.3.4.Prop.1]), and hence $r(f)$ finite. □

Proposition 1.4.6. *Let $f: X \to Y$ be a morphism of finite type between analytic affinoid adic spaces. Then equivalent are*
a) *f is proper*
b) *f is finite.*

Proof. b) implies a) by (1.4.5.ii), and a) implies b) by the subsequent Lemma (1.4.7). □

Lemma 1.4.7. *Let $f: X \to Y$ be a morphism of finite type between analytic affinoid adic spaces. Then, for every $y \in Y$ such that $X \times_Y \mathrm{Spa}\kappa(y) \to \mathrm{Spa}\kappa(y)$ is proper, there exists an open neighbourhood U of y such that $f^{-1}(U) \to U$ is finite. In particular, if f is proper then f is finite.*

Proof. We fix a $y \in Y$ such that $f_y: Z := X \times_Y \mathrm{Spa}\kappa(y) \to \mathrm{Spa}\kappa(y)$ is proper. Put $K := \kappa(y)^{\wedge}$ and $C := (\mathcal{O}_Z(Z), \mathcal{O}_Z^{+}(Z))$. Then C is of topologically finite type over K. Let I be an ideal of definition of K^{+}, and let C_0 be an open subring of C^{+} such that C_0 is of topologically finite type over K^{+} and C^{+} is the integral closure of C_0 in $C^{\flat}$. The topology of C_0 is adic and IC_0 is an ideal of definition. We consider the commutative diagram

$$(*) \qquad \begin{array}{ccc} Z & \xrightarrow{\lambda} & \mathrm{Spec}C_0/IC_0 \\ {\scriptstyle f_y}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{Spa}\kappa(y) & \xrightarrow[\mu]{} & \mathrm{Spec}K^{+}/I \end{array}$$

with $\lambda(z) = \{c \in C_0 \mid |c(z)| < 1\} \mathrm{mod} IC_0$ and $\mu(z) = \{c \in K^+ \mid |c(z)| < 1\} \mathrm{mod} I$. The mappings λ and μ are surjective. f_y satisfies the valuative criterion for properness (1.3.9). Then, using the diagram $(*)$, one can easily conclude that also g satisfies the valuative criterion for properness, i.e. g is proper (cf. [Hu$_4$, 3.12.5]). Then by [EGA IV.18.12.8] g is finite. This implies that $K^+ \to C_0$ is finite ([M, 28.P.Lemma]), and hence $K \to C$ is finite. Now we consider the commutative diagram

$$\begin{array}{ccc} C & \xleftarrow{\alpha} & B \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\delta} \\ K & \xleftarrow[\gamma]{} & A \end{array}$$

with $A = (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ and $B = (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$. We may assume that $A^\flat$ is a Tate ring. Since δ is of topologically finite type, there exists a quotient mapping $\varepsilon: A\langle T_1, \ldots, T_n\rangle \to B$ with $\delta = \varepsilon | A$. Put $s_i := \varepsilon(T_i) \in B^+$ $(i = 1, \ldots, n)$. Since $K^+ \to C^+$ is integral, there exists, for every $i \in \{1, \ldots, n\}$, a monic polynomial $p_i(S) \in K^+[S]$ with $p_i(\alpha(s_i)) = 0 \in C^+$. Let $\gamma_S: A^+[S] \to K^+[S]$ be the ring homomorphism induced by γ. Replacing Y by a suitable open affinoid neighbourhood U of y and X by $f^{-1}(U)$, we may assume that there exists, for every $i \in \{1, \ldots, n\}$, a monic polynomial $q_i \in A^+[S]$ such the coefficients of $\gamma_S(q_i) - p_i$ are elements of $(K^\flat)^{\circ\circ}$. Then, for every $i \in \{1, \ldots, n\}$, $\alpha(q_i(s_i)) = \gamma_S(q_i)(\alpha(s_i)) \in (C^\flat)^{\circ\circ}$. From this one can conclude by the subsequent Lemma (1.4.8) that, after replacing Y by a suitable open affinoid neighbourhood U of y and X by $f^{-1}(U)$, one has $q_i(s_i) \in (B^\flat)^{\circ\circ}$ for $i = 1, \ldots, n$. This implies that δ is finite ((1.4.3)). □

Lemma 1.4.8. *Let (A, A^+) and (B, B^+) be affinoid rings which are complete and Tate, and let $f: (A, A^+) \to (B, B^+)$ be a morphism such that B is of topologically finite type over A. Let b be an element of B and x a point of $\mathrm{Spa}(A, A^+)$ such that the image of b in $B \otimes_A k(x)$ is contained in $(B \otimes_A k(x))^{\circ\circ}$. Then there exists a rational subset U of $\mathrm{Spa}(A, A^+)$ such that $x \in U$ and the image of b in $B(g^{-1}(U))$ is contained in $B(g^{-1}(U))^{\circ\circ}$ where $g := \mathrm{Spa}(f): \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ and $B(g^{-1}(U)) := \mathcal{O}_{\mathrm{Spa}(B, B^+)}(g^{-1}(U))$.*

Proof. Let K be the quotient field of $A/\mathrm{supp}(x)$, and let $K_0 \subseteq K$ be the valuation ring of K induced by x. Let A_0 and B_0 be rings of definition of A and B such that $f(A_0) \subseteq B_0$ and $\mathrm{im}(A_0 \to K) \subseteq K_0$, and let $s \in A_0$ be a topologically nilpotent unit of A. Then $B \otimes_A k(x)$ is the completion of $B \otimes_A K$ where $B \otimes_A K$ is equipped with the I-adic topology such that $B_0K_0 := \mathrm{im}(B_0 \otimes_{A_0} K_0 \to B \otimes_A K)$ is a ring of definition and $s \cdot (B_0K_0)$ is an ideal of definition of B_0K_0. Let N be the neighbourhood of 0 in $B \otimes_A k(x)$ such that $s \cdot (B_0K_0)$ is the preimage of N under the morphism $B \otimes_A K \to (B \otimes_A K)^\wedge = B \otimes_A k(x)$. Choose a $m \in \mathbb{N}$ such that $b^m \in N$. Then the image of b^m in $B \otimes_A K$ is contained in $s \cdot (B_0K_0)$. Replacing b by b^m, we may assume that the image b' of b in $B \otimes_A K$ is contained in $s \cdot (B_0K_0)$. Write

$$b' = s \cdot \left(b_1 \otimes \frac{c_1}{d_1} + \ldots + b_n \otimes \frac{c_n}{d_n}\right) \tag{1}$$

with $b_1, \ldots, b_n \in B_0$ and $c_1, \ldots, c_n, d_1, \ldots, d_n \in A$ such that $\frac{c_i}{d_i} \in K_0$ for $i = 1, \ldots, n$. Choose a $\ell \in \mathbb{N}$ such that $1 \leq x(s^{-\ell} d_i)$ for $i = 1, \ldots, n$. Replacing c_i by $s^{-\ell} c_i$ and d_i by $s^{-\ell} d_i$, we may assume that $1 \leq x(d_i)$ for $i = 1, \ldots, n$. Then x is an element of the rational subset

$$U_1 := \bigcap_{i=1}^{n} R\left(\frac{c_i, 1}{d_i}\right) \subseteq \mathrm{Spa}(A, A^+).$$

Put $d := d_1 \cdot \ldots \cdot d_n \in A$. Then $B(g^{-1}(U_1))$ is the completion of the localization B_d where B_d is equipped with the f-adic topology such that $B' := B_0[\frac{c_i}{d_i}, \frac{1}{d_i} \mid i = 1, \ldots, n] \subseteq B_d$ is a ring of definition and $s \cdot B'$ is an ideal of definition of B'. Put

$$b'' := s \cdot \left(b_1 \cdot \tfrac{c_1}{d_1} + \ldots + b_n \cdot \tfrac{c_n}{d_n}\right) \in s \cdot B'. \tag{2}$$

Let $\bar{b}$ be the image of b in B_d, and let $\varphi : B_d \to B \otimes_A K$ be the natural ring homomorphism. Then by (1) and (2)

$$\varphi(b'') = b' = \varphi(\bar{b}),$$

and therefore

$$\varphi(\bar{b} - b'') = 0.$$

Since $\ker(\varphi) = \mathrm{supp}(x) \cdot B_d$, there exist $p_1, \ldots, p_m \in \mathrm{supp}(x)$ and $t_1, \ldots, t_m \in B_d$ with

$$b - b'' = p_1 t_1 + \ldots + p_m t_m. \tag{3}$$

Choose a $\ell \in \mathbb{N}$ such that $s^\ell t_i \in sB'$ for $i = 1, \ldots, m$. Replacing t_i by $s^\ell t_i$ and p_i by $\frac{p_i}{s^\ell}$, we may assume

$$t_i \in sB' \quad \text{for} \quad i = 1, \ldots, m. \tag{4}$$

Put

$$U := U_1 \cap \bigcap_{i=1}^{m} R\left(\frac{p_i, 1}{1}\right) \subseteq \mathrm{Spa}(A, A^+).$$

Since $p_i(x) = 0$ for $i = 1, \ldots, m$, we have $x \in U$. $B(g^{-1}(U))$ is the completion of B_d where B_d is equipped with the f-adic topology such that $B'' := B'[p_1, \ldots, p_m] \subseteq B_d$ is a ring of definition and $s \cdot B''$ is an ideal of definition of B''. By (3) and (4) we have $b - b'' \in sB''$, and then by (2)

$$b \in sB''.$$

This implies that the image of $\bar{b}$ in $B(g^{-1}(U))$ is contained in $B(g^{-1}(U))^{\circ\circ}$. □

Lemma 1.4.9. *Let*

$$\begin{array}{ccc} S' & \xrightarrow{r} & S \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ T' & \xrightarrow[h]{} & T \end{array}$$

be a cartesian diagram of adic spaces with T affinoid, h finite and $\mathcal{O}_{T'}(T')$ faithfully flat over $\mathcal{O}_T(T)$. Then if g' is finite then g is finite.

Proof. For every affinoid open subspace U of S, the ring $\mathcal{O}_{S'}(r^{-1}(U)) = \mathcal{O}_S(U) \otimes_{\mathcal{O}_T(T)} \mathcal{O}_{T'}(T')$ is finite over $\mathcal{O}_S(U)$ and carries the natural $\mathcal{O}_S(U)$-module topology (cf. proof of (1.4.5.i)). Hence the ring homomorphism $\mathcal{O}_S(U) \to \mathcal{O}_{S'}(r^{-1}(U))$ is a topological embedding ([Hu, 2.3.i and 2.4.ii]). Then, for every quasi-compact quasi-separated open subset V of S, we have $\mathcal{O}_{S'}(r^{-1}(V)) = \mathcal{O}_S(V) \otimes_{\mathcal{O}_T(T)} \mathcal{O}_{T'}(T')$ and the ring homomorphism $\mathcal{O}_S(V) \to \mathcal{O}_{S'}(r^{-1}(V))$ is a topological embedding.
Assume that g' is finite. Then S is quasi-compact and quasi-separated (since r is surjective and quasi-compact, and S' is quasi-compact and quasi-separated), and $\mathcal{O}_S(S)$ is finite over $\mathcal{O}_T(T)$ (since $\mathcal{O}_{S'}(S') = \mathcal{O}_S(S) \otimes_{\mathcal{O}_T(T)} \mathcal{O}_{T'}(T')$ is finite over $\mathcal{O}_{T'}(T')$ [EGA, IV.2.7.1]). Let Z be the adic space finite over T which corresponds to $\mathcal{O}_S(S)$ ((1.4.2)), and let $p: S \to Z$ be the natural T-morphism. Then $p': S' \to Z' := Z \times_T T'$ is an isomorphism. This implies that p is an isomorphism. In order to see this let us consider the cartesian diagram

$$(*) \qquad \begin{array}{ccc} S' & \xrightarrow{r} & S \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Z' & \xrightarrow[t]{} & Z \end{array}$$

p is injective (since p' is injective, t is surjective, and, for every $z' \in Z'$ and $s \in S$ with $t(z') = p(s)$, there exists a $s' \in S'$ with $z' = p'(s')$ and $s = r(s')$), p is surjective (since p' and t are surjective), and p is closed (since r is surjective, and p' and t are closed). Let U be a quasi-compact open subset of Z. Then $(*)$ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_S(p^{-1}(U)) \otimes_{\mathcal{O}_T(T)} \mathcal{O}_{T'}(T') = \mathcal{O}_{S'}(r^{-1}(p^{-1}(U))) & \xleftarrow{r^*} & \mathcal{O}_S(p^{-1}(U)) \\ {\scriptstyle p'^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ \mathcal{O}_Z(U) \otimes_{\mathcal{O}_T(T)} \mathcal{O}_{T'}(T') = \mathcal{O}_{Z'}(t^{-1}(U)) & \xleftarrow[t^*]{} & \mathcal{O}_Z(U) \end{array}$$

p^* is a topological embedding (since p'^*, t^*, r^* are topological embeddings) and p^* is bijective (since p'^* is bijective and $\mathcal{O}_{T'}(T')$ is faithfully flat over $\mathcal{O}_T(T)$). This shows that p is an isomorphism. □

1.5 Quasi-finite morphisms

Definition 1.5.1. Let $f: X \to Y$ be a morphism of adic spaces.
i) f is called *locally quasi-finite* if f is locally of weakly finite type and, for every $y \in Y$, $f^{-1}(y)$ is a discrete topological space.
ii) f is called *quasi-finite* if f is quasi-compact and locally quasi-finite.

Lemma 1.5.2. *Let $f: X \to Y$ be a morphism of analytic adic spaces which is weakly of finite type. Equivalent are*
a) *f is quasi-finite.*
b) *For every $y \in Y$, $f^{-1}(y)$ is a finite set.*
c) *For every maximal point y of Y, $f^{-1}(y)$ is a finite set.*
d) *For every $x \in X$, $k(x)^\wedge$ is finite over $k(f(x))^\wedge$.*
e) *For every maximal point x of X, $k(x)^\wedge$ is finite over $k(f(x))^\wedge$.*
f) *For every maximal point y of Y, the morphism $X \times_Y \operatorname{Spa}\kappa(y) \to \operatorname{Spa}\kappa(y)$ is finite.*
The equivalence of a), b), d) *holds without the assumption that X and Y are analytic.*

Proof. By (1.2.4) we may assume that Y is the spectrum of a complete affinoid field (K, K^+), X is the spectrum of a complete affinoid ring (A, A^+) over (K, K^+) such that A is of topologically finite type over K and $X \neq \emptyset$. We consider the commutative diagram

$$(*) \qquad \begin{array}{ccc} \operatorname{Spa}(A, A^\circ) & \xrightarrow{i} & \operatorname{Spa}(A, A^+) = X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \operatorname{Spa}(K, K^\circ) & \longrightarrow & \operatorname{Spa}(K, K^+) = Y \end{array}$$

By [BGR, 6.1.2.Cor.2] there exists, for some $d \in \mathbb{N}_0$, an injective finite K-algebra homomorphism $L_d := K\langle T_1, \ldots, T_d\rangle \longrightarrow A$. By [BGR, 6.3.4.Prop.1] A° is integral over L_d°. Thus we have the commutative diagram

$$(**) \qquad \begin{array}{ccc} \operatorname{Spa}(A, A^\circ) & \xrightarrow{h} & \operatorname{Spa}(L_d, L_d^\circ) \\ \downarrow{\scriptstyle g} & \swarrow & \\ \operatorname{Spa}(K, K^\circ) & & \end{array}$$

with h finite and surjective. The diagrams $(*)$ and $(**)$ show that each statement b), c), d), e) is equivalent to the statement $d = 0$.
For every $y \in Y$, $f^{-1}(y)$ is quasi-compact. Hence a) implies b). Conversely, b) implies $d = 0$ which gives a).
Obviously, f) implies e). Assume e) is satisfied.
Then $d = 0$, i.e. $L_d = K$. In order to show f) we may assume $K^+ = K^\circ$. Since A° is integral over K°, we have $A^+ = A^\circ$. Hence f is finite. □

Corollary 1.5.3. *Let*

$$\begin{array}{ccc} S & \xrightarrow{v} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[g]{} & Y \end{array}$$

be a cartesian diagram of adic spaces.
i) *Assume that one of the following conditions is satisfied*

a) *f is locally quasi-finite and locally of finite type.*

b) *f is locally quasi-finite and g is adic.*

Then u is locally quasi-finite.
ii) *If f is locally quasi-finite then, for every $x \in X$ and $t \in T$ with $f(x) = g(t)$, there exists a $s \in S$ with $x = v(s)$ and $t = u(s)$* (cf. (1.2.3.iii.d)).

Proof. i) Let $s \in S$ be given, and let $t \in T$, $x \in X$ and $y \in Y$ be the image points of s. Then we have the commutative diagram

$$\begin{array}{ccc} k(s)^\wedge & \xleftarrow{\beta} & k(x)^\wedge \\ {\scriptstyle\alpha}\uparrow & & \uparrow \\ k(t)^\wedge & \longleftarrow & k(y)^\wedge \end{array}$$

By the construction of the fibre product in the proof of (1.2.2), the subfield of $k(s)^\wedge$ generated by $\mathrm{im}(\alpha) \cup \mathrm{im}(\beta)$ is dense in $k(s)^\wedge$. Hence if $k(x)^\wedge$ is finite over $k(y)^\wedge$ then $k(s)^\wedge$ is finite over $k(t)^\wedge$. Therefore by (1.5.2) u is locally quasi-finite.
ii) Put $y := f(x) = g(t) \in Y$. Since $k(x)^\wedge$ is finite over $k(y)^\wedge$ ((1.5.2)), there exists a commutative diagram of fields

$$\begin{array}{ccc} K & \xleftarrow{\beta'} & k(x)^\wedge \\ {\scriptstyle\alpha'}\uparrow & & \uparrow{\scriptstyle f^*} \\ k(t)^\wedge & \xleftarrow[g^*]{} & k(y)^\wedge \end{array}$$

and a valuation ring K^+ of K such that K is finite over $k(t)^\wedge$ and K^+ extends the valuation rings $V(x)$ and $V(t)$ of $k(x)^\wedge$ and $k(t)^\wedge$ which are induced by the valuation rings $k(x)^+$ and $k(t)^+$ of $k(x)$ and $k(t)$.
If $V(t)$ is microbial ((1.1.4)) then also K^+ is microbial (since K is finite over $k(t)^\wedge$). We equip K with a topology, namely if t is an analytic point of T (and hence $V(t)$ is microbial) then we equip K with the valuation topology of K^+ and if t

is a non-analytic point of T (and hence the topology of $k(t)^\wedge$ is discrete) then we equip K with the discrete topology. Then (K, K^+) is an affinoid field, and α and β induce continuous ring homomorphisms of affinoid rings $\beta' : \kappa(x) = (k(x), k(x)^+) \to (K, K^+)$ and $\alpha' : \kappa(t) = (k(t), k(t)^+) \to (K, K^+)$.
Composing $\mathrm{Spa}(\beta') : \mathrm{Spa}(K, K^+) \to \mathrm{Spa}\kappa(x)$ and $\mathrm{Spa}(\alpha') : \mathrm{Spa}(K, K^+) \to \mathrm{Spa}\kappa(t)$ with the natural morphisms $\mathrm{Spa}\kappa(x) \to X$ and $\mathrm{Spa}\kappa(t) \to T$ ((1.1.8)), we obtain morphisms $v' : \mathrm{Spa}(K, K^+) \to X$ and $u' : \mathrm{Spa}(K, K^+) \to T$ with $f \circ v' = g \circ u'$. Let $w : \mathrm{Spa}(K, K^+) \to S$ be the morphism induced by u' and v', and let c be the closed point of $\mathrm{Spa}(K, K^+)$. Then $s := w(c) \in S$ satisfies $x = v(s)$ and $t = u(s)$. □

Proposition 1.5.4. *Let $f : X \to Y$ be a morphism of analytic adic spaces which is locally of weakly finite type.*
Then equivalent are
a) *f is locally quasi-finite.*
b) *For every maximal point x of X, $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,f(x)}$.*
If f is locally of finite type then a) *and* b) *are equivalent to*
c) *For every maximal point x of X, there exist open subspaces $U \subseteq X$ and $V \subseteq Y$ such that $x \in U$, $f(U) \subseteq V$ and $f : U \to V$ is finite.*

Proof. Obviously b) or c) implies a) ((1.5.2)). Let f be locally quasi-finite. In order to show b) and c) we may assume that X and Y are affinoid. Put $A = (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ and $B = (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$. Let x_0 be a maximal point of X. Then $y_0 := f(x_0)$ is a maximal point of Y ((1.1.10.iv)).
1) Let us first assume that B is of topologically finite type over A.
Then by (1.5.2.a, f) and (1.4.7) there exists an open affinoid neighbourhood V of y_0 such that $f^{-1}(V) \to V$ is finite. Let $x_0, \ldots, x_n$ be the points of $f^{-1}(y_0)$. Then every x_i is a maximal point of X. Hence $\mathcal{O}_{Y,y_0} = \mathcal{O}_Y(V)\mathcal{O}_X(f^{-1}(V)) = \varinjlim_{x \in W} \mathcal{O}_X(f^{-1}(W)) = \prod_{i=0}^{n} \mathcal{O}_{X,x_i}$. Therefore $\mathcal{O}_{X,x_0}$ is finite over $\mathcal{O}_{Y,y_0}$. Hence c) and b) are satisfied.
2) Assume that $B^\flat$ is of topologically finite type over $A^\flat$.
Let C be a ring of integral elements of $B^\flat$ such that $(B^\flat, C)$ is of topologically finite type over A. Put $Z := \mathrm{Spa}(B^\flat, (B^\flat)^\circ)$ and $X' := \mathrm{Spa}(B^\flat, C)$. Then we have natural Y-morphisms $i : Z \to X$ and $j : Z \to X'$. Since i is a topological embedding, Z is quasi-finite over Y. The image of Z under j contains all maximal points of X', and, for every $z \in Z$, $k(j(z))^\wedge \xrightarrow{\sim} k(z)^\wedge$. Hence by (1.5.2) X' is quasi-finite over Y. Let z_0 be the maximal point of Z with $x_0 = i(z_0)$. Then $x_0' := j(z_0)$ is a maximal point of X'. Since $\mathcal{O}_{X',x_0'}$ is finite over $\mathcal{O}_{Y,y_0}$ (by part 1) of the proof) and $\mathcal{O}_{X',x_0'} \xrightarrow{\sim} \mathcal{O}_{Z,z_0}$ and $\mathcal{O}_{X,x_0} \xrightarrow{\sim} \mathcal{O}_{Z,z_0}$, we obtain that $\mathcal{O}_{X,x_0}$ is finite over $\mathcal{O}_{Y,y_0}$. Hence b) is satisfied. □

Proposition 1.5.5. *Let $f : X \to Y$ be a morphism between analytic adic spaces which is locally of finite type. Equivalent are*
a) *f is quasi-finite and proper*
b) *f is finite*

Proof. Every finite morphism is proper ((1.4.5.ii)) and quasi-finite ((1.5.2)). Assume that f is quasi-finite and proper. Let y be a point of Y. We have to show that there exists an open neighbourhood U of y such that $f^{-1}(U) \to U$ is finite. By (1.5.2) the set $f^{-1}(y)$ is finite and, for every $x \in f^{-1}(y)$, $k(x)^\wedge$ is finite over $k(y)^\wedge$. Therefore there exists an open affinoid neighbourhood V of y and a finite morphism $h: W \to V$ such that $\mathcal{O}_W(W)$ is faithfully flat over $\mathcal{O}_Y(V)$ and, for every $w \in W$ lying over y and every $z \in Z := f^{-1}(V) \times_V W$ lying over w, the field $k(z)^\wedge$ is finite and purely inseparable over $k(w)^\wedge$.

$$\begin{array}{ccc} Z & \longrightarrow & f^{-1}(V) \\ {\scriptstyle g}\downarrow & & \downarrow \\ W & \xrightarrow[h]{} & V \end{array}$$

We have

(1) For every $w \in h^{-1}(y)$, there exists an open neighbourhood L of w in W such that $g^{-1}(L) \to L$ is finite.

Proof. We fix a $w \in h^{-1}(y)$. Let $z_1, \ldots, z_n$ be the points of $g^{-1}(w)$. Let G be the set of the generalizations of w in W, and, for every $i \in \{1, \ldots, n\}$, let G_i be the set of the generalizations of z_i in Z. Then $G_i \subseteq g^{-1}(G)$. We have

(2) i) For every $i \in \{1, \ldots, n\}$, G_i is open and closed in $g^{-1}(G)$.

ii) For every $i, j \in \{1, \ldots, n\}$ with $i \neq j$, $G_i \cap G_j = \emptyset$.

iii) $g^{-1}(G) = \bigcup_{i=1}^{n} G_i$.

Proof. Since g is specializing, we have $g^{-1}(G) = \bigcup_{i=1}^{n} G_i$. Hence it suffices to show that $G_i \cap G_j = \emptyset$ for $i \neq j$. Assume that there exist $i \neq j$ with $G_i \cap G_j \neq \emptyset$. We fix a $z \in G_i \cap G_j$, $w := g(z) \in W$. Let $(k(z), A_i)$ and $(k(z), A_j)$ be the valuation rings of Z with centers z_i and z_j ((1.3.6)). Then $A_i \cap k(w) = A_j \cap k(w)$ ((1.3.6)). Since $k(z)^\wedge (\simeq k(z_i)^\wedge)$ is purely inseparable over $k(w)^\wedge (\simeq k(w)^\wedge)$, we obtain $A_i = A_j$. Then by (1.3.7) $z_i = z_j$, contradiction. This shows (2).

Since G_i is open in $g^{-1}(G)$ (by (2i)), we can choose, for every $i = \{1, \dots, n\}$, an open subset K_i' of Z with $K_i' \cap g^{-1}(G) = G_i$. Let K_i be an open affinoid subset of K_i' such that $z_i \in K_i$ and $(\mathcal{O}_K(K_i), \mathcal{O}_K^+(K_i))$ is of topologically finite tpye over $(\mathcal{O}_W(W), \mathcal{O}_W^+(W))$. Since G_i is closed in $g^{-1}(G)$ (by (2i)) and $G_i = K_i \cap g^{-1}(G)$, there exists by (1.4.7) and (1.2.4) an open neighbourhood L_i of w in W such that $g: K_i \cap g^{-1}(L_i) \to L_i$ is finite. Since $\emptyset = G_i \cap G_j = (K_i \cap K_j) \cap g^{-1}(G)$ for $i \neq j$ (by (2ii)), there exists an open neighbourhood L' of w in W such that $K_i \cap K_j \cap g^{-1}(L') = \emptyset$ for $i \neq j$. Put $M := L' \cap \bigcap_{i=1}^{n} L_i'$. Then $g: (\bigcup_{i=1}^{n} K_i) \cap g^{-1}(M) \to M$ is finite. Since $g^{-1}(G) = \bigcup_{i=1}^{n} G_i \subseteq \bigcup_{i=1}^{n} K_i$ (by (2iii)), there exists an open neighbourhood L of w in M with $g^{-1}(L) \subset \bigcup_{i=1}^{n} K_i$. Then $g: g^{-1}(L) \to L$ is finite. This proves (1).

By (1) there exists an open neighbourhood S of $h^{-1}(w)$ in W such that $g^{-1}(S) \to S$ is finite. Since h is a closed mapping, there exists an open affinoid neighbourhood U of y in V with $h^{-1}(U) \subseteq S$. Then $g^{-1}(h^{-1}(U)) \to h^{-1}(U)$ is finite. This implies by (1.4.9) that $f^{-1}(U) \to U$ is finite. □

Proposition 1.5.6. *Let $f: X \to Y$ be a morphism of analytic adic spaces. We assume that f is locally of finite type, X is quasi-separated and Y is quasi-compact and quasi-separated. Then equivalent are*
a) f is locally quasi-finite and partially proper.
b) For every $x \in X$, there exist open subspaces U and V of X and Y such that the closure $\overline{\{x\}}$ of $\{x\}$ in X is contained in U, the closure $\overline{\{f(x)\}}$ of $\{f(x)\}$ in Y is contained in V, $f(U) \subset V$ and $f: U \to V$ is finite.

Proof. If b) is satisfied then f is locally quasi-finite ((1.5.2)) and partially proper ((1.3.9) and (1.4.5 n)).
Assume that a) is satisfied. Let $x_0 \in X$ be given. In order to check that b) is satisfied for x_0 we may assume that x_0 is a maximal point of X. Then $y_0 := f(x_0)$ is a maximal point of Y ((1.1.10.iv)). Let $\overline{\{x_0\}}$ be the closure of $\{x_0\}$ in X, and let $\overline{\{y_0\}}$ be the closure of $\{y_0\}$ in Y. By (1.3.13) there exists a quasi-compact open subset W' of X with $\overline{\{x_0\}} \subseteq W'$. Let $x_0, x_1, \dots, x_n$ be the maximal points of W' which lie over y_0 ((1.5.2)). Then $\overline{\{x_i\}} \cap \{x_j\} = \emptyset$ for $i \neq j$ ((1.1.10.i)) and $f^{-1}(\overline{\{y_0\}}) \cap W' = \bigcup_{i=1}^{n} (\overline{\{x_i\}} \cap W')$ ((1.1.10.v)). Hence $\overline{\{x_0\}}$ is open in $f^{-1}(\overline{\{y_0\}}) \cap W'$.
Thus there exists a quasi-compact open subset W of X with $\overline{\{x_0\}} = f^{-1}(\overline{\{y_0\}}) \cap W$.
We will show

(1) There exists an open neighbourhood V of $\overline{\{y_0\}}$ in Y such that $f: W \cap f^{-1}(V) \to V$ is proper.

Then by (1.5.5) $f: W \cap f^{-1}(V) \to V$ is finite, and we see that b) is satisfied for x_0. Thus it remains to show (1). In (1.3.11) we defined the topological space W_v, and by (1.3.12) W_v is spectral. We have

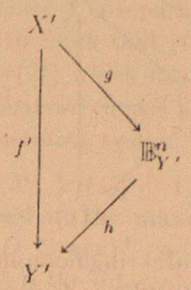
(2) The set $S := \{(x, A) \in W_v \mid$ there exists a $y \in Y$ such that y is a center of (x, A) on Y and no $t \in f^{-1}(y) \cap W$ is a center of (x, A) on $W\}$ is pro-constructible in W_v.

Proof. Let $Y = \bigcup_{i=1}^{m} Y_i$ be a covering of Y by open affinoid subspaces Y_i. For every $i \in \{1, \ldots, m\}$, put $S_i := \{(x, A) \in W_v \mid f_v(x, A)$ has a center on Y_i and (x, A) has no center on $f^{-1}(Y_i) \cap W\}$. Then $S = \bigcup_{i=1}^{m} S_i$ ((1.3.6.ii)). Hence it suffices to show that every S_i is pro-constructible in W_v. We fix a $i \in \{1, \ldots, m\}$. Let $f^{-1}(Y_i) \cap W = \bigcup_{j=1}^{k} W_j$ be a covering of $f^{-1}(Y_i) \cap W$ by open affinoid subspaces W_j such that, for every $j \in \{1, \ldots, k\}$, the ring homomorphism $(\mathcal{O}_{Y_i}(Y_i), \mathcal{O}^+_{Y_i}(Y_i)) \to (\mathcal{O}_X(W_j), \mathcal{O}^+_X(W_j))$ is of topologically finite type. In particular, there exists a finite subset E_j of $\mathcal{O}^+_X(W_j)$ such that $\mathcal{O}^+_X(W_j)$ is the integral closure of $\mathcal{O}^+_{Y_i}(Y_i)[E_j \cup \mathcal{O}_X(W_j)^{\circ\circ}]$ in $\mathcal{O}_X(W_j)$. Put $T := \{(x, A) \in W_v \mid (x, A)$ has a center on $Y_i\}$ and $T_j := \{(x, A) \in W_v \mid (x, A)$ has a center on $W_j\}$ $(j = 1, \ldots, k)$. Then $T_j = \{(x, A) \in T \mid x \in W_j$ and $c(x) \in A$ for every $c \in E_j\}$ ((1.3.6.ii)). By (1.3.6.ii) and (1.3.12) T is pro-constructible in W_v and T_j is constructible in T. Hence $S_i = T - \bigcup_{j=1}^{k} T_j$ is pro-constructible in W_v. This proves (2).

Let $\mathfrak{U}$ denote the set of all quasi-compact open subsets U of Y with $\overline{\{y_0\}} \subset U$. Since $\overline{\{y_0\}}$ is closed under generalizations in Y ((1.1.10.i)), we have $\overline{\{y_0\}} = \bigcap_{U \in \mathfrak{U}} U$. Put $S(\overline{\{y_0\}}) := \{(x, A) \in S \mid x \in f^{-1}(\overline{\{y_0\}}) \cap W\}$ and, for every $U \in \mathfrak{U}$, $S(U) := \{(x, A) \in S \mid x \in f^{-1}(U) \cap W\}$. By (2) S is a pro-constructible subset of W_v, and by (1.3.12) $S(\overline{\{y_0\}})$ and $S(U)$ are pro-constructible subsets of S. Since f is partially proper and $f^{-1}(\overline{\{y_0\}}) \cap W (= \{x_0\})$ is closed in X, we have $S(\overline{\{y_0\}}) = \emptyset$ ((1.3.9)). Then $S(V) = \emptyset$ for some $V \in \mathfrak{U}$, since $S(\overline{\{y_0\}}) = \bigcap_{U \in \mathfrak{U}} S(U)$. Now (1.3.9) gives that $f: W \cap f^{-1}(V) \to V$ is proper (notice that W is quasi-compact). This proves (1) and finishes the proof of (1.5.6). □

Proposition 1.5.7. *Let $f: X \to Y$ be a morphism of rigid analytic varieties, and let $r(f): r(X) \to r(Y)$ be the associated morphism of adic spaces. Equivalent are*
a) $f: X \to Y$ is locally quasi-finite (i.e., for every $y \in Y$, $\dim f^{-1}(y) \leq 0$).
b) $r(f): r(X) \to r(Y)$ is locally quasi-finite.

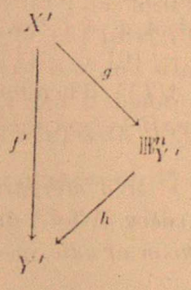
Proof. We may assume that X and Y are affinoid. Let y be a maximal point of $r(Y)$ with $r(f)^{-1}(y) \neq \emptyset$. By the subsequent Lemma (1.5.8) there exist a locally closed affinoid rigid analytic subvariety Y' of Y and a $n \in \mathbb{N}_0$ such that $y \in r(Y') \subseteq r(Y)$ and the morphism $f' : X' := X \times_Y Y' \to Y'$ has a factorization



where $h : \mathbb{B}^n_{Y'} := \mathrm{Sp}\mathcal{O}_{Y'}(Y')\langle T_1, \ldots, T_n\rangle \to Y'$ is the natural morphism and g is finite and surjective.

If f is locally quasi-finite then $n = 0$ and hence by (1.5.2) $r(f)$ is locally quasi-finite. Conversely, if $r(f)$ is locally quasi-finite then f is locally quasi-finite by (1.5.2 a), b)). □

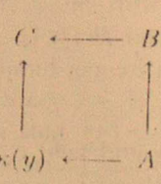
Lemma 1.5.8. *Let $f : X \to Y$ be a morphism of analytic affinoid adic spaces which is of finite type, and let y be a maximal point of Y. Then there exist a locally closed affinoid adic subspace Y' of Y and an integer $n \geq -1$ such that $y \in Y'$ and the morphism $f' : X' := X \times_Y Y' \to Y'$ induced by f has a factorization*



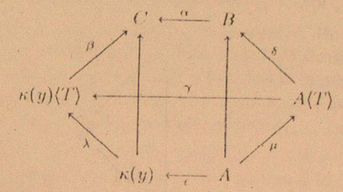
where $h : \mathbb{B}^n_{Y'} = \mathrm{Spa}(\mathcal{O}_{Y'}(Y'), \mathcal{O}^+_{Y'}(Y'))\langle T_1, \ldots, T_n\rangle \to Y'$ is the natural morphism and g is finite and surjective ($\mathbb{B}^{-1}_{Y'} := \emptyset$).

Proof Put $A = (\mathcal{O}_Y(Y), \mathcal{O}^+_Y(Y))$, $B = (\mathcal{O}_X(X), \mathcal{O}^+_X(X))$ and $C = B \otimes_A \kappa(y)$,

(*)



The proof of [T, 4.4] shows that if we replace Y by a suitable open affinoid neighbourhood U of y and X by $f^{-1}(U)$ then one can extend $(*)$ to a commutative diagram of continuous morphisms of affinoid rings



where λ and μ are the natural ring homomorphisms, $T = (T_1, \ldots, T_n)$ and $\gamma(T_i) = T_i$ for $i = 1, \ldots, n$, and $\beta\colon \kappa(y)\langle T\rangle \to C$ is finite and injective. Since δ is of topologically finite type, there exists a quotient mapping $\sigma\colon A\langle T\rangle\langle S_1, \ldots, S_m\rangle \to B$ with $\sigma|A\langle T\rangle = \delta$. Put $s_i := \sigma(S_i) \in B^+$. Since the ring homomorphism $\beta\colon \kappa(y)\langle T\rangle^+ \to C^+$ is integral, there exists, for every $i \in \{1, \ldots, m\}$, a monic polynomial $p_i \in \kappa(y)\langle T\rangle^+[Z]$ with $p_i(\alpha(s_i)) = 0$. Let $\gamma_Z\colon A\langle T\rangle^+[Z] \to \kappa(y)\langle T\rangle^+[Z]$ be the ring homomorphism induced by γ. Replacing Y by an open affinoid neighbourhood U of y and X by $f^{-1}(U)$, there exist monic polynomials $q_1, \ldots, q_m \in A\langle T\rangle^+[Z]$ such that the coefficients of $\gamma_Z(q_i) - p_i$ are contained in $k(y)\langle T\rangle^{\circ\circ}$. Then $\alpha(q_i(s_i)) = \gamma_Z(q_i)(\alpha(s_i)) \in (C^\circ)^{\circ\circ}$ for $i = 1, \ldots, m$. Therefore, after replacing Y by a suitable open affinoid neighbourhood U of y and X by $f^{-1}(U)$, we may assume that $q_i(s_i) \in (B^\circ)^{\circ\circ}$ for $i = 1, \ldots, m$ ((1.4.8)). This implies that δ is finite ((1.4.3)). Put $\mathfrak{p} = \mathrm{supp}(y) = \ker(\varepsilon\colon A^\circ \to k(y))$. Substituting A by $A/\mathfrak{p}$ and B be $B/\mathfrak{p}B^\circ$, we may assume that ε is injective. Then also γ is injective. Since β is injective, there exists a prime ideal $\mathfrak{q}$ of C° with $\beta^{-1}(\mathfrak{q}) = \{0\} \subseteq \kappa(y)\langle T\rangle^\circ$. Then $(\alpha \circ \delta)^{-1}(\mathfrak{q}) = \gamma^{-1}(\beta^{-1}(\mathfrak{q})) = \{0\} \subseteq A\langle T\rangle^\circ$. Hence there exists a prime ideal $\mathfrak{r}$ of B° with $\delta^{-1}(\mathfrak{r}) = \{0\} \subseteq A\langle T\rangle^\circ$. Therefore, since δ is injective, we obtain that $\mathrm{Spec}B^\circ \to \mathrm{Spec}A\langle T\rangle^\circ$ and $\mathrm{Spa}B \to \mathrm{Spa}A\langle T\rangle$ are surjective. □

Lemma 1.5.9. *Let $f\colon X \to Y$ be a morphism of rigid analytic varieties which is locally quasi-finite (i.e., for every $y \in Y$, $\dim f^{-1}(y) \leq 0$), and let $r(f)\colon r(X) \to r(Y)$ be the associated morphism of adic spaces. Equivalent are*

(a) *f is separated and, for every open affinoid subspace Y_0 of Y, there exists an admissible open covering $(X_i)_{i\in I}$ of $f^{-1}(Y_0)$ by open affinoid subspaces X_i of $f^{-1}(Y_0)$ such that, for every $i \in I$, there exists an open affinoid subspace X_i' of $f^{-1}(Y_0)$ such that $X_i \overset{\subset\subset}{Y_0} X_i'$.*

(b) *f is partially proper ((1.3.19.i)).*

(c) *$r(f)$ is partially proper.*

(If the valuation of the ground field is discrete then the equivalence of (b) and (c) is already stated in (1.3.19.iii).)

Proof. (a) $\Rightarrow$ (b): trivial.

(b) $\Rightarrow$ (c): We may assume that $Y = \mathrm{Sp}A$ is affinoid and there exists an admissible covering $X = \bigcup_{i \in I} X_i$ of X by affinoid open subspaces $X_i = \mathrm{Sp}B_i$ such that, for every $i \in I$, there exists an affinoid open subspace $Z_i = \mathrm{Sp}C_i$ of X such that $X_i \underset{Y}{\subset\subset} Z_i$. The morphism $r(f)$ is separated since f is separated. Hence in order to show that $r(f)$ is partially proper it suffices to check that $r(f)$ satisfies (1.3.8.b).

Let (x, D) be a valuation ring of $r(X)$ which has a center on $r(Y)$. We show that (x, A) has a center on $r(X)$. For that we choose a $i \in I$ such that $x \in r(X_i)$. Since $X_i \underset{Y}{\subset\subset} Z_i$, there exist an affinoid generating system $c_1, \ldots, c_n$ of C_i over A such that, for every $j \in \{1, \ldots, n\}$ and $z \in X_i$, $|c_j(z)| < 1$. Then, for every $j \in \{1, \ldots, n\}$, $|c_j(x)| < 1$, i.e., $c_j(x)$ is contained in the maximal ideal $\mathfrak{m}$ of $k(x)^+$. Since D is a valuation ring of $k(x)$ with $D \subseteq k(x)^+$, we have $\mathfrak{m} \subseteq D$. Hence, for every $j \in \{1, \ldots, n\}$, $c_j(x) \in D$. Since C_i° is the integral closure of the topological closure of $A^\circ[c_1, \ldots, c_n]$ in C_i, we obtain that, for every $c \in C_i^\circ$, $c(x) \in D$. This means that (x, D) has a center on $\mathrm{Spa}(C_i, C_i^\circ) = r(Z_i) \subseteq r(X)$.

(c) $\Rightarrow$ (a): We may assume that Y is affinoid. We fix a maximal point x of $r(X)$. Put $y := r(f)(x) \in r(Y)$. By (1.5.6) and (1.5.7) there exist open subspaces $U \subseteq r(X)$ and $V \subseteq r(Y)$ such that $\overline{\{x\}} \subseteq U$, $\overline{\{y\}} \subseteq V$, $r(f)(U) \subseteq V$ and $r(f)\colon U \to V$ is finite. Since y is a maximal point of $r(Y)$, the subsequent lemma says $\overline{\{y\}} = \bigcap_{i \in I} N_i$ where every N_i is a rational subset of $r(Y)$. Hence there exists a rational subset V' of $r(Y)$ with $\overline{\{y\}} \subseteq V' \subseteq V$. Replacing V and U by V' and $U \cap r(f)^{-1}(V')$, we may assume that V and U are affinoid.

Let A and B be the complete Tate algebras with $r(Y) = \mathrm{Spa}(A, A^\circ)$ (i.e., $Y = \mathrm{Sp}A$) and $U = \mathrm{Spa}(B, B^\circ)$. Since $r(f)\colon r(X) \to r(Y)$ is partially proper, every valuation ring of $r(X)$ which has a center on $r(Y)$ has also a center on $r(X)$ (by (1.3.9)). By construction of U the closure $\overline{\{x\}}$ of $\{x\}$ in $r(X)$ is contained in U. Hence every valuation ring of $r(X)$ whose support is x and which has a center on $r(Y)$ has also a center on U. This means that if $\tilde{A}$ and $\tilde{B}$ denote the images of A° and B° in $k(x)$ then, for every valuation ring D of $k(x)$ with $\tilde{A} \subseteq D \subseteq k(x)^+$, we have $\tilde{B} \subseteq D$, or equivalently, if $\overset{\approx}{A}$ and $\overset{\approx}{B}$ denote the images of A° and B° in the residue field K of $k(x)^+$ then, for every valuation ring E of K with $\overset{\approx}{A} \subseteq E$, we have $\overset{\approx}{B} \subseteq E$. Hence

(1) $$\overset{\approx}{B} \text{ is integral over } \overset{\approx}{A}.$$

Let $b_1, \ldots, b_n \in B^\circ$ be an affinoid generating system of B over A and let $s \in A$ be a topologically nilpotent unit. By (1) there exist monic polynomials $p_1, \ldots, p_n \in A^\circ[T]$ with $|p_i(b_i)(x)| < 1$ for $i = 1, \ldots, n$. Now we consider the subsets $G := \{p_i(b_i) | i = 1, \ldots, n\} \cup \{sb_i | i = 1, \ldots, n\} \subseteq B^\circ$ and $W := \{u \in U \,|\, |g(u)| < 1 \text{ for every } g \in G\} \subseteq U$. Then G is an affinoid generating system of B over A and W is a closed constructible subset of U. Since $x \in W$, there exists by the subsequent lemma a rational subset U_x of U such that $\overline{\{x\}} \subset U_x \subseteq W$. Let Z and Z_x be the open affinoid subspaces of X with $U = r(Z)$ and $U_x = r(Z_x)$. Since $U_x \subseteq W$, we

have $Z_x \underset{Y}{\subset\subset} Z$. Furthermore, $\{Z_x | x$ maximal point of $r(X)\}$ is an admissible open covering of X. Indeed, since $\overline{\{x\}} \subseteq U_x = r(Z_x)$ for every maximal point x of $r(X)$, we have $r(X) = \bigcup \{r(Z_x) | x$ maximal point of $r(X)\}$. □

Lemma 1.5.10. *Let A be an affinoid ring which is Tate and let x be a maximal point of* Spa$A$. *Let $\mathcal{N}$ be the set of all rational subsets of* Spa$A$ *which contain the closure $\overline{\{x\}}$ of $\{x\}$ in* Spa$A$. *For every $N \in \mathcal{N}$, let $\overline{N}$ be the closure of N in* Spa$A$. *Then*

$$\overline{\{x\}} = \bigcap_{N \in \mathcal{N}} N = \bigcap_{N \in \mathcal{N}} \overline{N}.$$

Proof. First we show

(1) a) Let $(P_i)_{i \in I}$ be a family of pro-constructible subsets of Spa$A$ which are closed under generalizations. Then $\overline{P} = \bigcap_{i \in I} \overline{P_i}$ with $P := \bigcap_{i \in I} P_i$.

b) Let M be a rational subset of Spa$A$. Then there exists a sequence of rational subsets $M_1 \supseteq M_2 \supseteq \ldots$ with $\overline{M} = \bigcap_{n \in \mathbb{N}} M_n$.

Proof. a) Let $x \in \bigcap_{i \in I} \overline{P_i}$ be given. For every $i \in I$, we choose a $p_i \in P_i$ such that x is a specialization of p_i. By (1.1.10.i) there exists a maximal point p of Spa$A$ such that, for every $i \in I$, p_i is a specialization of p. Since every P_i is closed under generalizations, $p \in \bigcap_{i \in I} P_i = P$. Hence $x \in \overline{P}$.

b) Let $a_1, \ldots, a_k, b$ be elements of $A^\flat$ such that $A^\flat = a_1 A^\flat + \ldots + a_k A^\flat$ and $M = \{x \in \mathrm{Spa}A \mid |a_i(x)| \leq |b(x)| \text{ for } i = 1, \ldots, k\}$, and let $s \in A^\flat$ be a topologically nilpotent unit. For every $n \in \mathbb{N}$, put $M_n := \{x \in \mathrm{Spa}A \mid |s a_i^n(x)| \leq |b^n(x)| \text{ for } i = 1, \ldots, k\}$. Then $(M_n)_{n \in \mathbb{N}}$ is a decreasing sequence of rational subsets of Spa$A$. It is easily seen that every specialization of a point of M is contained in $\bigcap_{n \in \mathbb{N}} M_n$. Hence $\overline{M} \subset \bigcap_{n \in \mathbb{N}} M_n$.

Conversely, if x is an element of $\bigcap_{n \in \mathbb{N}} M_n$ then the maximal point of Spa$A$ which is a generalization of x is contained in M, and hence $x \in \overline{M}$. This concludes the proof of (1).

Since by assumption in (1.5.10) x is a maximal point of Spa$A$, there exists a set $\mathcal{M}$ of rational subsets of Spa$A$ with

$$\{x\} = \bigcap_{M \in \mathcal{M}} M.$$

Then by (1.a) $\overline{\{x\}} = \bigcap_{M \in \mathcal{M}} \overline{M}$. Hence (1.b) implies $\overline{\{x\}} = \bigcap_{N \in \mathcal{N}} N$. Then again by (1.a) $\overline{\{x\}} = \bigcap_{N \in \mathcal{N}} \overline{N}$. □

1.6 Definition of unramified, smooth and étale morphisms

Definition 1.6.1. Let A, B be complete f-adic rings and $A \to B$ a ring homomorphism of topologically finite type. A *universal A-derivation* of B is a continuous A-derivation $d_{B|A}: B \to \Omega_{B|A}$ from B to a complete topological B-module $\Omega_{B|A}$ such that, for any continuous A-derivation $d: B \to M$ from B to a complete topological B-module M, there exists a unique continuous B-linear map $g: \Omega_{B|A} \to M$ with $d = g \circ d_{B|A}$.

In the following we equip every module M finitely generated over a complete f-adic ring A with its natural A-module topology ([Hu, §2]). Then every A-linear map $M \to N$ from M to a topological A-module N is continuous, M is complete and every submodule of M is closed ([Hu, 2.3 and 2.4]).

(1.6.2) Constructions of $(\Omega_{B|A}, d_{B|A})$. Let A, B be complete f-adic rings and $A \to B$ a ring homomorphism of topologically finite type.

i) Let $M_1, \ldots, M_n$ be finite subsets of A with $M_i \cdot A$ open in A for $i = 1, \ldots, n$ and let I be an ideal of $A\langle X_1, \ldots, X_n\rangle_{M_1, \ldots, M_n} =: C$ such that $B = C/I$. Let P be a set of generators of I.

Let Ω be the free C-module of rank n with generators $dX_1, \ldots, dX_n$, let $d: C \to \Omega$ be the mapping with $d(s) = \sum_{i=1}^{n} \frac{\partial s}{\partial X_i} dX_i$, and let $\Omega_{B|A}$ be the quotient of Ω by the C-submodule generated by $P \cdot \Omega \cup \{d(p) \mid p \in P\}$. Then there exists a mapping $d_{B|A}: B \to \Omega_{B|A}$ such that

$$\begin{array}{ccc} C & \xrightarrow{d} & \Omega \\ \downarrow & & \downarrow \\ B & \xrightarrow[d_{B|A}]{} & \Omega_{B|A} \end{array}$$

commutes, and $(\Omega_{B|A}, d_{A|B})$ is a universal A-derivation of B. In particular, $\Omega_{B|A}$ is a finitely generated B-module and if $B = C$ then $\Omega_{B|A}$ is a free B-module with basis $d_{B|A}(X_1), \ldots, d_{B|A}(X_n)$.

ii) Let $I \subset B \otimes_A B$ denote the kernel of the ring homomorphism $B \otimes B \to B$, $a \otimes b \mapsto ab$. Then I is a finitely generated ideal and hence $\Omega_{B|A} := I/I^2$ a finitely generated B-module. Let $d_{B|A}: B \to \Omega_{B|A}$ be the mapping induced by $b \mapsto 1 \otimes b - b \otimes 1$. Then $(\Omega_{B|A}, d_{B|A})$ is a universal A-derivation of B.

iii) If B is finite over A then $d_{B|A}: B \to \Omega_{B|A}$ is a universal A-derivation of the *ring* B (i.e., without topology).

Proof. i) is easily checked.
ii) Let N be a finite subset of B such that the subring $A[N]$ is dense in B. We show that I is generated by $L := \{1\hat{\otimes}n - n\hat{\otimes}1 \mid n \in N\}$. Let J be the ideal of $B\hat{\otimes}B$ generated by L. Then $J \subseteq I$, and since J is closed in $B\hat{\otimes}B$ it is enough to show that J is dense I. The formula $1\hat{\otimes}ab - ab\hat{\otimes}1 = (1\hat{\otimes}a)(1\hat{\otimes}b - b\hat{\otimes}1) + (b\hat{\otimes}1)(1\hat{\otimes}a - a\hat{\otimes}1)$ (with $a, b \in B$) shows that $\{1\hat{\otimes}n - n\hat{\otimes}1 \mid n \in A[N]\}$ is contained in J. Let $i \in I$ and a neighbourhood U of 0 in $B\hat{\otimes}B$ be given. We choose a neighbourhood R of 0 in $B\hat{\otimes}B$ with $R = -R$ and $R + R \subseteq U$, a neighbourhood S of 0 in B with $S\hat{\otimes}1 \subseteq R$, a neighbourhood T of i in $i + R$ which is mapped into S under the continuous ring homomorphism $B\hat{\otimes}B \to B$, $a\hat{\otimes}b \mapsto ab$, and $x_1, \ldots, x_m, y_1, \ldots, y_m \in A[N]$ with $\sum_{k=1}^{m} x_k\hat{\otimes}y_k \in T$. Put $j := \sum_{k=1}^{m} (x_k\hat{\otimes}1)(1\hat{\otimes}y_k - y_k\hat{\otimes}1) \in J$. We have $\sum_{k=1}^{m} x_k y_k \in S$, and hence $(\sum_{k=1}^{m} x_k y_k)\hat{\otimes}1 \subseteq R$. Then the formula $j = \sum_{k=1}^{m} x_k\hat{\otimes}y_k - (\sum_{k=1}^{m} x_k y_k)\hat{\otimes}1$ gives $j \in \sum_{k=1}^{m} x_k\hat{\otimes}y_k + R$ which implies $j \in i + R + R \subseteq i + U$. This shows that I is generated by L. Now (ii) can be proved as in the algebraic setting (see for example [I, 1.1.3]).
iii) follows from ii). □

The following proposition is easily checked.

Proposition 1.6.3. *Let A, B, C be complete f-adic rings and $A \to B$ and $B \to C$ ring homomorphisms of topologically finite type. Then*
i) *One has a natural exact sequence of finitely generated C-modules*

$$\Omega_{B|A}\hat{\otimes}_B C \longrightarrow \Omega_{C|A} \longrightarrow \Omega_{C|B} \longrightarrow 0.$$

ii) *If $B \to C$ is open and surjective with kernel I then there is a natural exact sequence of finitely generated C-modules*

$$I/I^2 \longrightarrow \Omega_{B|A}\hat{\otimes}_B C \longrightarrow \Omega_{C|A} \longrightarrow 0.$$

iii) *Let A' be a complete f-adic ring and $A \to A'$ an adic ring homomorphism, and put $B' := B\hat{\otimes}_A A'$. Then $\Omega_{B'|A'} = \Omega_{B|A}\hat{\otimes}_B B'$.*
iv) *Let s be an element of B and T a finite subset of B such that $T \cdot B$ is open in B. In* [Hu, §1] *there is defined the localization $B(\frac{T}{s})$ of B and its completion $B\langle\frac{T}{s}\rangle$. Let $d : B(\frac{T}{s}) \to \Omega_{B|A}\hat{\otimes}_B B(\frac{T}{s})$ be the derivation extending $d_{B|A} : B \to \Omega_{B|A}$. Then d is continuous if we equip $\Omega_{B|A}\hat{\otimes}_B B(\frac{T}{s})$ with its natural $B(\frac{T}{s})$-module topology. The completion $B(\frac{T}{s}) \to (\Omega_{B|A}\hat{\otimes}_B B(\frac{T}{s}))\hat{} = \Omega_{B|A}\hat{\otimes}_B B\langle\frac{T}{s}\rangle$ of d is a universal A-derivation of $B\langle\frac{T}{s}\rangle$.*

Proof. i) is easily checked.
ii) Let N be a finite subset of B such that the subring $A[N]$ is dense in B. We show that I is generated by $L := \{1\otimes n - n\otimes 1 \mid n \in N\}$. Let J be the ideal of $B\otimes B$ generated by L. Then $J \subseteq I$, and since J is closed in $B\otimes B$ it is enough to show that J is dense I. The formula $1\otimes ab - ab\otimes 1 = (1\otimes a)(1\otimes b - b\otimes 1) + (b\otimes 1)(1\otimes a - a\otimes 1)$ (with $a, b \in B$) shows that $\{1\otimes n - n\otimes 1 \mid n \in A[N]\}$ is contained in J. Let $i \in I$ and a neighbourhood U of 0 in $B\otimes B$ be given. We choose a neighbourhood R of 0 in $B\otimes B$ with $R = -R$ and $R + R \subseteq U$, a neighbourhood S of 0 in B with $S\otimes 1 \subseteq R$, a neighbourhood T of i in $i + R$ which is mapped into S under the continuous ring homomorphism $B\otimes B \to B$, $a\otimes b \mapsto ab$, and $x_1, \ldots, x_m, y_1, \ldots, y_m \in A[N]$ with $\sum_{k=1}^{m} x_k\otimes y_k \in T$. Put $j := \sum_{k=1}^{m} (x_k\otimes 1)(1\otimes y_k - y_k\otimes 1) \in J$. We have $\sum_{k=1}^{m} x_k y_k \in S$, and hence $(\sum_{k=1}^{m} x_k y_k)\otimes 1 \subseteq R$. Then the formula $j = \sum_{k=1}^{m} x_k\otimes y_k - (\sum_{k=1}^{m} x_k y_k)\otimes 1$ gives $j \in \sum_{k=1}^{m} x_k\otimes y_k + R$ which implies $j \in i + R + R \subseteq i + U$. This shows that I is generated by L. Now (ii) can be proved as in the algebraic setting (see for example [I, I.1.3]).
iii) follows from ii). □

The following proposition is easily checked.

Proposition 1.6.3. *Let A, B, C be complete f-adic rings and $A \to B$ and $B \to C$ ring homomorphisms of topologically finite type. Then*
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$$\Omega_{B|A}\otimes_B C \longrightarrow \Omega_{C|A} \longrightarrow \Omega_{C|B} \longrightarrow 0.$$

ii) *If $B \to C$ is open and surjective with kernel I then there is a natural exact sequence of finitely generated C-modules*

$$I/I^2 \longrightarrow \Omega_{B|A}\otimes_B C \longrightarrow \Omega_{C|A} \longrightarrow 0.$$

iii) *Let A' be a complete f-adic ring and $A \to A'$ an adic ring homomorphism, and put $B' := B\otimes_A A'$. Then $\Omega_{B'|A'} = \Omega_{B|A}\otimes_B B'$.*
iv) *Let s be an element of B and T a finite subset of B such that $T \cdot B$ is open in B. In* [Hu, §1] *there is defined the localization $B(\frac{T}{s})$ of B and its completion $B\langle\frac{T}{s}\rangle$. Let $d: B(\frac{T}{s}) \to \Omega_{B|A}\otimes_B B(\frac{T}{s})$ be the derivation extending $d_{B|A}: B \to \Omega_{B|A}$. Then d is continuous if we equip $\Omega_{B|A}\otimes_B B(\frac{T}{s})$ with its natural $B(\frac{T}{s})$-module topology. The completion $B\langle\frac{T}{s}\rangle \to (\Omega_{B|A}\otimes_B B(\frac{T}{s}))^{\wedge} = \Omega_{B|A}\otimes_B B\langle\frac{T}{s}\rangle$ of d is a universal A-derivation of $B\langle\frac{T}{s}\rangle$.*

Let X, Y be adic spaces and $f: X \to Y$ a morphism locally of weakly finite type. Let $\Delta: X \to X \times_Y X =: Z$ be the diagonal morphism and $\mathcal{I} \subseteq \mathcal{O}_Z$ the kernel of $\mathcal{O}_Z \to \Delta_* \mathcal{O}_X$. We call the $\mathcal{O}_X$-module $\Omega_{X|Y} := \Delta^*(\mathcal{I}) = \mathcal{I} \otimes_{\mathcal{O}_Z} \mathcal{O}_X$ the sheaf of differentials of X over Y. If U and V are open affinoid subspaces of X and Y such that $f(U) \subseteq V$ and $\mathcal{O}_X(U)$ is of topologically finite type over $\mathcal{O}_Y(V)$ then $\Omega_{X|Y}|U$ is the $(\mathcal{O}_X|U)$-module associated with the finite $\mathcal{O}_X(U)$-module $\Omega_{\mathcal{O}_X(U)|\mathcal{O}_Y(V)}$.

Lemma 1.6.4. *Let $f: X \to Y$ be a locally quasi-finite morphism between analytic adic spaces. Then*
i) *For every maximal point $x \in X$, $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,f(x)}$ and the $\mathcal{O}_{Y,f(x)}$-derivation $d: \mathcal{O}_{X,x} \to (\Omega_{X|Y})_x$ induced by the differential $d_{X|Y}: \mathcal{O}_X \to \Omega_{X|Y}$ is a universal $\mathcal{O}_{Y,f(x)}$-derivation of $\mathcal{O}_{X,x}$.*
ii) *If f is flat (i.e., for every $x \in X$, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$) then, for every morphism of adic spaces $Y' \to Y$, the projection $f': X' := X \times_Y Y' \to Y'$ is flat.*

Proof. i) follows from (1.6.2.iii) and the proof of (1.5.4).
ii) Let x' be a maximal point of X', and let $y' \in Y'$, $x \in X$ and $y \in Y$ be the image points of x'. Then $\mathcal{O}_{X',x'}$ is a direct factor of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'}$ (see proof of (1.5.4)). Hence $\mathcal{O}_{X',x'}$ is flat over $\mathcal{O}_{Y',y'}$. Then by (1.1.10.iii), for every $x \in X'$, $\mathcal{O}_{X',x}$ is flat over $\mathcal{O}_{Y',f'(x)}$. □

Definition 1.6.5. i) A morphism $f: X \to Y$ between adic spaces is called *unramified* (resp. *smooth*, resp. *étale*) if f is locally of finite type (resp. locally of finite presentation, resp. locally of finite presentation) and if, for any affinoid ring A, any ideal I of $A^\flat$ with $I^2 = \{0\}$ and any morphism $\mathrm{Spa}A \to Y$, the mapping $\mathrm{Hom}_Y(\mathrm{Spa}A, X) \to \mathrm{Hom}_Y(\mathrm{Spa}A/I, X)$ is injective (resp. surjective, resp. bijective).
ii) A morphism $f: X \to Y$ between adic spaces is called unramified (resp. smooth, resp. étale) at a point $x \in X$ if there exist an open neighbourhood U of x in X and an open subspace V of Y such that $f(U) \subseteq V$ and $f: U \to V$ is unramified (resp. smooth, resp. étale).

Example 1.6.6. i) Let $f: \mathrm{Spa}A \to \mathrm{Spa}B$ be a morphism of affinoid adic spaces, and assume that the topology of $B^\flat$ is discrete and the topology of $A^\flat$ is complete. Then f is unramified (resp. smooth, resp. étale) if and only if the topology of $A^\flat$ is discrete, A^+ is the integral closure in $A^\flat$ of a finitely generated B^+-subalgebra of $A^\flat$ and the morphism of schemes $\mathrm{Spec}A^\flat \to \mathrm{Spec}B^\flat$ is unramified (resp. smooth, resp. étale).
ii) Let A be a noetherian complete f-adic ring and A^+ a ring of integral elements of A. By (1.4.2) and (1.4.4) the category of adic spaces finite over $\mathrm{Spa}(A, A^+)$ is canonically equivalent to the category of schemes finite over $\mathrm{Spec}A$. Also the category of adic spaces which are finite and étale over $U := \mathrm{Spa}(A, A^+)$ is canonically equivalent to the category of schemes which are finite and étale over $X := \mathrm{Spec}A$. Indeed, one knows that a finite morphism of schemes $Y \to X$ is étale if, for any finite morphism of schemes $Y' \to X$ and any closed subscheme $Y'_0 \subseteq Y'$ defined by an ideal I of $\mathcal{O}_{Y'}(Y')$ with $I^2 = \{0\}$, the mapping $\mathrm{Hom}_X(Y', Y) \to \mathrm{Hom}_X(Y'_0, Y)$ is

bijective, and similarly one can prove that a finite morphism of adic spaces $V \to U$ is étale if, for any finite morphism of adic spaces $V' \to U$ and any closed adic subspace $V_0' \subseteq V'$ defined by an ideal I of $\mathcal{O}_{V'}(V')$ with $I^2 = \{0\}$, the mapping $\mathrm{Hom}_U(V, V') \to \mathrm{Hom}_U(V, V_0')$ is bijective.

The following propositions (1.6.7), (1.6.8) and (1.6.9) can be proved similarly as for schemes (cf. [EGA], [I]).

Proposition 1.6.7. i) *Open embeddings are étale, locally closed embeddings are unramified, and, for any affinoid ring A and finite subsets $T_1, \ldots, T_n$ of $A^\flat$ with $T_i \cdot A^\circ$ open in $A^\flat$ for $i = 1, \ldots, n$, the natural morphism $\mathrm{Spa}A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n} \to \mathrm{Spa}A$ is smooth.*
ii) *The composition of two étale (resp. smooth, resp. unramified) morphisms is étale (resp. smooth, resp. unramified).*
iii) *Let $f: X \to Y$ and $g: Y \to Z$ be two morphisms. If $g \circ f$ is unramified then f is unramified. If $g \circ f$ is étale (resp. smooth) and g unramified then f is étale (resp. smooth).*
iv) *If $X \to Y$ is an étale (resp. smooth, resp. unramified) morphism and $Y' \to Y$ an arbitrary morphism then $X \times_Y Y' \to Y'$ is étale (resp. smooth, resp. unramified).*
v) *A morphism $f: X \to Y$ is étale (resp. smooth, resp. unramified) if and only if f is étale (resp. smooth, resp. unramified) at any point $x \in X$.*

Proposition 1.6.8. *Let X, Y be adic spaces and $f: X \to Y$ a morphism locally of finite type. Then the following conditions are equivalent*

a) *f is unramified*

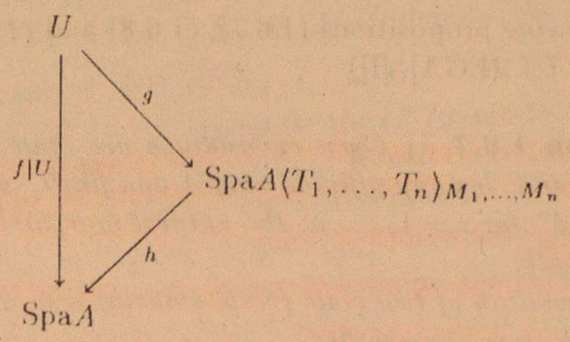
b) $\Omega_{X|Y} = 0$

c) *The diagonal morphism $\Delta(f): X \to X \times_Y X$ is an open embedding.*

d) *For any open affinoid subspace U of X such that $f(U)$ is contained in an open affinoid subspace of Y the restriction $f|U: U \to Y$ factors into $U \xrightarrow{g} Z \xrightarrow{h} Y$ where g is a closed embedding into an affinoid adic space Z and h is an étale morphism.*

Proposition 1.6.9. *Let $X \to Y$ be a smooth morphism between adic spaces. Then*
i) *$\Omega_{X|Y}$ is a locally free $\mathcal{O}_X$-module.*
ii) *Let Z be a closed subspace of X defined by a finitely generated sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. Then Z is étale (resp. smooth) over Y at a point $x \in Z$ if and only if there exist generators $s_1, \ldots, s_n$ of the ideal $\mathcal{I}_x \subseteq \mathcal{O}_{X,x}$ such that $d_{X|Y}(s_1), \ldots, d_{X|Y}(s_n) \in (\Omega_{X|Y})_x$ form (resp. can be extended to) a basis of the $\mathcal{O}_{X,x}$-module $(\Omega_{X|Y})_x$.*
iii) *Let Z be an adic space locally of finite type over Y and $f: X \to Z$ a Y-morphism. Then f is étale (resp. smooth) at a point $x \in X$ if and only if the natural $\mathcal{O}_{X,x}$-linear mapping $d: (\Omega_{Z|Y} \otimes_{\mathcal{O}_Z} \mathcal{O}_X)_x \to (\Omega_{X|Y})_x$ is bijective (resp. has a left-inverse).*

Corollary 1.6.10. *Let A be an affinoid ring and $f: X \to \mathrm{Spa}A$ a morphism of adic spaces. f is smooth if and only if, for every $x \in X$, there exists a commutative diagram*



(∗)

where U is an open neighbourhood of x in X, $M_1, \ldots, M_n$ are finite subsets of $A^\triangleright$ with $M_i \cdot A^\triangleright$ open in $A^\triangleright$ for $i = 1, \ldots, n$, h is the natural morphism and g is étale. If A is Tate and f is smooth then, for every $x \in X$, there exists a commutative diagram (∗) with $M_1 = \ldots = M_n = \{1\}$.

Proof. Put $Y := \mathrm{Spa}A$. Assume that f is smooth. By (1.6.9.i), for every $x \in X$, there exist an open affinoid neighbourhood U of x in X and $s_1, \ldots, s_n \in \mathcal{O}_X(U)$ such that $\Omega_{X|Y}|U$ is a free $\mathcal{O}_X|U$-module with basis $d_{X|Y}(s_1), \ldots, d_{X|Y}(s_n) \in \Omega_{X|Y}(U)$. Choose finite subsets $M_1, \ldots, M_n$ of $A^\triangleright$ such that $M_i \cdot A^\triangleright$ is open in $A^\triangleright$ and $M_i \cdot s_i \in \mathcal{O}_X^+(U)$ for $i = 1, \ldots, n$. Then there exists a commutative diagram (∗) with $g^*(T_i) = s_i$ for $i = 1, \ldots, n$ ([Hu, 3.5.i]). By (1.6.9.iii) and (1.6.2.i) g is étale. Assume that $A^\triangleright$ is Tate. Let s be a topologically nilpotent unit of $A^\triangleright$. Choose a $\ell \in \mathbb{N}$ such that $s^\ell \cdot s_i \in \mathcal{O}_X^+(U)$ for $i = 1, \ldots, n$. Replacing s_i by $s^\ell \cdot s_i$, we may assume that $s_i \in \mathcal{O}_X^+(U)$ for $i = 1, \ldots, n$. Then we can choose $M_1 = \ldots = M_n = \{1\}$. □

1.7 Étale morphisms

Proposition 1.7.1. *Let A be an affinoid ring (which is not necessarily complete), and let $s: X \to Y := \mathrm{Spa}A$ be a morphism between affinoid adic spaces. Then the following conditions are equivalent*

i) *s is étale.*

ii) *There exist finite subsets $T_1, \ldots, T_n$ of $A^\triangleright$ with $T_i \cdot A^\triangleright$ open in $A^\triangleright$ for $i = 1, \ldots, n$ and there exist $f_1, \ldots, f_n \in A^\triangleright\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}$ such that the following holds: If we put $B := A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}$ and $I := f_1 B^\triangleright + \ldots + f_n B^\triangleright$ then the image of the determinant $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n} \in B^\triangleright$ in $B^\triangleright/I$ is a unit of $B^\triangleright/I$ and X is Y-isomorphic to $\mathrm{Spa}B/I$.*

iii) *There exist finite subsets $T_1, \ldots, T_n$ of $A^\triangleright$ with $T_i \cdot A^\triangleright$ open in $A^\triangleright$ for $i = 1, \ldots, n$ and there exist $f_1, \ldots, f_n \in A^\triangleright[X_1, \ldots, X_n]$ such that the following holds: If we put $B := A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}$ and $I := f_1 B^\triangleright + \ldots + f_n B^\triangleright$ then the image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n} \in A^\triangleright[X_1, \ldots, X_n]$ in $B^\triangleright/I$ is a unit of $B^\triangleright/I$ and X is Y-isomorphic to $\mathrm{Spa}B/I$.*

Proof. (i) $\Longrightarrow$ (ii): We choose finite subsets $T_1, \ldots, T_m$ of $A^\flat$ with $T_i \cdot A^\flat$ open for $i = 1, \ldots, m$ and an ideal J of $C := A\langle X_1, \ldots, X_m\rangle_{T_1,\ldots,T_m}$ such that X is Y-isomorphic to $\mathrm{Spa}C/J$. The natural mapping of $C^\flat/J$-modules $e: J/J^2 \to \Omega_{C^\flat|A^\flat} \otimes_{C^\flat} C^\flat/J$ is bijective (cf. (1.6.3.ii), (1.6.8), (1.6.9.ii)), and $\Omega_{C^\flat|A^\flat} \otimes_{C^\flat} C^\flat/J$ is a free $C^\flat/J$-module of rank m ((1.6.2.i)). Let $f_1, \ldots, f_m$ be elements of J such that $e(f_1), \ldots, e(f_m)$ is a basis of $\Omega_{C^\flat|A^\flat} \otimes_{C^\flat} C^\flat/J$. ($f_1, \ldots, f_m \in J/J^2$ denote the residue classes of $f_1, \ldots, f_m$). Then the image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,m} \in C^\flat$ in $C^\flat/J$ is a unit of $C^\flat/J$, ((1.6.2.i)). Furthermore, $\bar{f}_1, \ldots, \bar{f}_m$ generate the $C^\flat/J$-module J/J^2. Then by Nakayama's Lemma $J \cdot C^\flat_{1+J} = \{f_1, \ldots, f_m\} \cdot C^\flat_{1+J}$. Hence there exists a $j \in J$ with $J \cdot C^\flat_S = \{f_1, \ldots, f_m\} \cdot C^\flat_S$ where S is the multiplicative system $\{(1+j)^n \mid n \in \mathbb{N}_0\}$ of $C^\flat$. Then $V := \{x \in \mathrm{Spa}C \mid 1 \le v_x(1+j)\} = \mathrm{Spa}C\langle\frac{1}{1+j}\rangle$ is a rational subset of $\mathrm{Spa}C$ with $J \cdot \mathcal{O}_{\mathrm{Spa}C}|V = \{f_1, \ldots, f_m\} \cdot \mathcal{O}_{\mathrm{Spa}C}|V$. Since V contains the subspace $\mathrm{Spa}C/J$ of $\mathrm{Spa}C$, we obtain $X = \mathrm{Spa}C/J = \mathrm{Spa}C\langle\frac{1}{1+j}\rangle/J \cdot C\langle\frac{1}{1+j}\rangle = \mathrm{Spa}C\langle\frac{1}{1+j}\rangle/(f_1, \ldots, f_m)$. Let X_{m+1} be a variable, and put $f_{m+1} = 1 - (1+j)X_{m+1}$. Then $C\langle\frac{1}{1+j}\rangle = C\langle X_{m+1}\rangle/(f_{m+1})$, and hence $X = \mathrm{Spa}C\langle X_{m+1}\rangle/(f_1, \ldots, f_{m+1}) = \mathrm{Spa}B/I$ with $B := A\langle X_1, \ldots, X_{m+1}\rangle_{T_1,\ldots,T_m,\{1\}}$ and $I := f_1 B^\flat + \ldots + f_{m+1} B^\flat$. The image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,m+1} = -(1+j)\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,m} \in B^\flat$ in $B^\flat/I$ is a unit of $B^\flat/I$.
(ii) $\Longrightarrow$ (iii): We need some preparations which are given in the following points (1) to (4).

(1) Let A be a complete f-adic ring and $f_i = \sum_{\nu \in \mathbb{N}_0^n} a_{i,\nu} X^\nu$, $i = 1, \ldots, n$ elements of $A[[X_1, \ldots, X_n]]$ such that, for every $i = 1, \ldots, n$, there exists a bounded subset K of A with $a_{i,\nu} \in K^{|\nu|} = K \cdot \ldots \cdot K$ ($|\nu|$-fold product) for every $\nu \in \mathbb{N}_0^n$, $a_{i,(0,\ldots,0)} = 0$ and $\det(a_{i,e_j})_{i,j=1,\ldots,n} \in A^*$ (e_j denotes the j-th unit vector $(0, \ldots, \overset{j}{1}, \ldots, 0)$). Then there exists a neighbourhood U of 0 in $A^{\times n} = A \times \ldots \times A$ such that, for every $i = 1, \ldots, n$ and $x \in U$, the infinite series $\sum_{\nu \in \mathbb{N}_0^n} a_{i,\nu} x^\nu$ converges in A (the limit is denoted by $f_i(x)$) and $\{(f_1(x), \ldots, f_n(x)) \mid x \in U'\}$ is a neighbourhood of 0 in $A^{\times n}$ for any neighbourhood U' of 0 in U.

Proof. Let $(b_{ij})_{i,j=1,\ldots,n} \in \mathrm{Mat}(n \times n, A)$ be the inverse matrix of $(a_{i,e_j})_{i,j=1,\ldots,n}$. It is enough to prove the assertion for $f'_1, \ldots, f'_n \in A[[X_1, \ldots, X_n]]$ with $f'_i := \sum_{j=1}^n b_{ij} f_j$. Hence we may assume $a_{i,e_i} = 1$ and $a_{i,e_j} = 0$ for $i \neq j$. By [B, III.4.4 Prop. 5] there exist $g_i = \sum_{\nu \in \mathbb{N}_0^n} b_{i,\nu} X^\nu \in A[[X_1, \ldots, X_n]]$, $i = 1, \ldots, n$ with $g_i(0) = 0$ and $g_i(f_1, \ldots, f_n) = X_i = f_i(g_1, \ldots, g_n)$ for $i = 1, \ldots, n$. Let L be a bounded subset of A with $a_{i,\nu} \in L^{|\nu|}$ for $i = 1, \ldots, n$, $\nu \in \mathbb{N}_0^n$, and put $M := (L \cup \{1\})^2$. Then M is a bounded subset of A, and $a_{i,\nu} \in M^{|\nu|-1}$ for $i = 1, \ldots, n$, $\nu \in \mathbb{N}_0^n$ (put $M^{-1} = \{0\}$). By induction on $|\nu|$ one can prove $b_{i,\nu} \in M^{|\nu|}$ for $i = 1, \ldots, n$, $\nu \in \mathbb{N}_0^n$. Let P be a neighbourhood of 0 in A such that $(P^n \mid n \in \mathbb{N})$ is a zero sequence (i.e. to every neighbourhood W of 0 in A there exists a $k \in \mathbb{N}$ with $P^\ell \subseteq W$ for $\ell > k$),

let S be a neighbourhood of 0 in A with $S \cdot L \subseteq P$, $S + S \subseteq S$, $S \cdot S \subseteq S$, and let T be a neighbourhood of 0 in A with $T \cdot M \subseteq S \cap P$. For every $x \in S^{\times n}$ and $i = 1, \ldots, n$, the series $\sum_{\nu \in \mathbb{N}_0^n} a_{i,\nu} x^\nu$ converges in A (since $a_{i,\nu} \in L^{|\nu|}$), and, for every $x \in T^{\times n}$, the series $\sum_{\nu \in \mathbb{N}_0^n} b_{i,\nu} x^\nu$ converges in A (since $b_{i,\nu} \in M^{|\nu|}$). Let $f_i(x)$ (resp. $g_i(x)$) denote the limit of $\sum_{\nu \in \mathbb{N}_0^n} a_{i,\nu} x^\nu$ (resp. $\sum_{\nu \in \mathbb{N}_0^n} b_{i,\nu} x^\nu$). Then, for evey $x = (x_1, \ldots, x_n) \in T^{\times n}$ and $i = 1, \ldots, n$, $g_i(x) \in S$ and $f_i(g_1(x), \ldots, g_n(x)) = x_i$ (the latter since $f_i(g_1, \ldots, g_n) = X_i$). Hence $\{(f_1(x), \ldots, f_n(x)) \mid x \in S^{\times n}\}$ is a neighbourhood of 0 in $A^{\times n}$.

(2) Let A be a f-adic ring, $T_1, \ldots, T_n$ finite subsets of A with $T_i \cdot A$ open in A, B a complete f-adic A-algebra such that the structure morphism $f: A \to B$ is adic, $g: A\langle X\rangle_T := A\langle X_1, \ldots, X_n\rangle_{T_1, \ldots, T_n} \to B$ an adic A-algebra homomorphism, $f_1, \ldots, f_n$ elements of $A\langle X\rangle_T$ such that the image of the determinant $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n} \in A\langle X\rangle_T$ in B is a unit of B, and U a neighbourhood of 0 in B. Then there exists a neighbourhood V of 0 in B such that, for every $(v_1, \ldots, v_n) \in V^{\times n}$, there exists a continuous A-algebra homomorphism $h: A\langle X\rangle_T \to B$ with $h(f_i) = g(f_i) + v_i$ and $h(X_i) - g(X_i) \in U$ for $i = 1, \ldots, n$.

Proof. Since $f: A \to B$ is adic, $f(T_i) \cdot B$ is open in B for $i = 1, \ldots, n$, and since $g: A\langle X\rangle_T \to B$ is adic, $f(t) \cdot g(X_i)$ is power-bounded in B for every $i \in \{1, \ldots, n\}$, $t \in T_i$. Hence by the universal property ([Hu, 3.3.i]) there exist continuous ring homomorphisms $r: A\langle X\rangle_T \to B\langle X\rangle_{f(T)} := B\langle X_1, \ldots, X_n\rangle_{f(T_1), \ldots, f(T_n)}$ and $s: B\langle X\rangle_{f(T)} \to B\langle X\rangle_{f(T)}$ such that r extends f and $r(X_i) = X_i$ for $i = 1, \ldots, n$, and $s(b) = b$ for every $b \in B$ and $s(X_i) = g(X_i) + X_i$ for $i = 1, \ldots, n$. We put $m_i := s(r(f_i) - g(f_i)) \in B\langle X\rangle_{f(T)}$ for $i = 1, \ldots, n$. Then $\det(\frac{\partial m_i}{\partial X_j}(0))_{i,j=1,\ldots,n} \in B$ is a unit of B (since $B^* \ni g(\det \frac{\partial f_i}{\partial X_j}) = \det \frac{\partial r(f_i)}{\partial X_j}(g(X_1), \ldots, g(X_n)) = \det \frac{\partial (r(f_i) - g(f_i))}{\partial X_j}(g(X_1), \ldots, g(X_n)) = \det \frac{\partial m_i}{\partial X_j}(0))$ and $m_1(0) = \ldots = m_n(0) = 0$. If we write $m_i = \sum_{\nu \in \mathbb{N}_0^n} b_{i,\nu} X^\nu$ with $b_{i,\nu} \in B$ then, for every $i = 1, \ldots, n$, there exists a bounded subset K of B with $b_{i,\nu} \in K^{|\nu|}$ for any $\nu \in \mathbb{N}_0^n$. (That is true for any element of $B\langle X\rangle_{f(T)}$.) Let W be a neighbourhood of 0 in B such that $W \subseteq U$ and $w \cdot x$ is power-bounded in B for any $w \in W$ and $x \in \bigcup_{i=1}^n f(T_i)$. Then, for any $i = 1, \ldots, n$ and $x \in W^{\times n}$, the series $\sum_{\nu \in \mathbb{N}_0^n} b_{i,\nu} x^\nu$ converges to an element of B which we denote by $m_i(x)$. By (1) there exists a neighbourhood V of 0 in B with $V^{\times n} \subseteq \{(m_1(x), \ldots, m_n(x)) \mid x \in W^{\times n}\}$.

Let $(v_1, \ldots, v_n) \in V^{\times n}$ be given, and choose a $w = (w_1, \ldots, w_n) \in W^{\times n}$ with $v_i = m_i(w)$ for $i = 1, \ldots, n$. For any $i \in \{1, \ldots, n\}$ and $t \in T_i$, $f(t) \cdot (g(X_i) + w_i)$ is

power-bounded in B. Hence there exists a continuous A-algebra homomorphism $h: A\langle X\rangle_T \to B$ with $h(X_i) = g(X_i) + w_i$ for $i = 1, \ldots, n$. We have $h(f_i) = r(f_i)(g(X_1)+w_1, \ldots, g(X_n)+w_n) = (r(f_i) - g(f_i))(g(X_1)+w_1, \ldots, g(X_n)+w_n) + g(f_i) = s(r(f_i) - g(f_i))(w_1, \ldots, w_n) + g(f_i) = m_i(w) + g(f_i) = v_i + g(f_i)$.

(3) Let A be an affinoid ring, B a complete affinoid ring of topologically finite type over A, and $b_1, \ldots, b_n \in B^\flat$ a system of topological generators of B over A (i.e., there exist a quotient mapping $h: A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n} \to B$ over A with $b_i = h(X_i)$ for $i = 1, \ldots, n$ ($T_1, \ldots, T_n$ are finite subsets of $A^\flat$ with $T_i \cdot A^\flat$ open in $A^\flat$)). Then there exists a neighbourhood U of 0 in $B^\flat$ such that every continuous ring homomorphism $f: B \to B$ over A with $f(b_i) \in b_i + U$ for $i = 1, \ldots, n$ is an isomorphism. More precisely, for every neighbourhood V of 0 in $B^\flat$, there exists a neighbourhood U of 0 in $B^\flat$ such that every continuous ring homomorphism $f: B \to B$ over A with $f(b_i) \in b_i + U$ for $i = 1, \ldots, n$ is an isomorphism with $f^{-1}(b_i) \in b_i + V$ for $i = 1, \ldots, n$.

Proof. The assertion is trivial if the topology of $B^\flat$ is discrete. Hence we may assume that $B^\flat$ is noetherian ((1.1.1)). Let a neighbourhood V of 0 in $B^\flat$ be given. Let A_0 be a ring of definition of $A^\flat$ with $A_0 \subseteq A^+$, and let $I \subseteq A_0$ be an ideal of definition of A_0. There exists a ring of definition B_0 of $B^\flat$ such that $A_0[t \cdot b_i \mid i = 1, \ldots, n, t \in T_i]$ is a dense subring of B_0, and B^+ is the integral closure of $A^+ \cdot B_0$ in $B^\flat$, and $B^\flat = A^\flat \cdot B_0[b_1, \ldots, b_n]$. We choose a $k \in \mathbb{N}$ such that $U := I^k \cdot B_0$ satisfies $U \subseteq V$ and $T_i \cdot U \subseteq I \cdot B_0$ for $i = 1, \ldots, n$. Let $f: B \to B$ be a continuous ring homomorphism over A with $f(b_i) \in b_i + U$ for $i = 1, \ldots, n$. We show that f is an isomorphism with $f^{-1}(b_i) \in b_i + U$ for $i = 1, \ldots, n$. For $i = 1, \ldots, n$ and $t \in T_i$, we have $f(t \cdot b_i) = t \cdot f(b_i) \subseteq t \cdot (b_i + U) \subseteq t \cdot b_i + t \cdot U \subseteq t \cdot b_i + I \cdot B_0$. Since $A_0[t \cdot b_i \mid i = 1, \ldots, n, t \in T_i]$ is dense in B_0, we obtain $f(b) \in b + I \cdot B_0$ for every $b \in B_0$. Hence $f(b) \in b + I^{n+1} \cdot B_0$ for every $n \in \mathbb{N}_0$ and every $b \in I^n \cdot B_0$, i.e. $f(I^n \cdot B_0) \subseteq I^n \cdot B_0$ and the induced mapping $I^n \cdot B_0 / I^{n+1} \cdot B_0 \to I^n \cdot B_0 / I^{n+1} \cdot B_0$ is the identity. We conclude from [B, III.2.9 Cor. 3] that $f(B_0) = B_0$ and hence $f(U) = U$. Since $f(b_i) \in b_i + U$, there exists a $c_i \in U$ with $f(b_i) = b_i + f(c_i)$, i.e. $b_i = f(b_i - c_i)$. This implies that $f: B^\flat \to B^\flat$ is surjective (since $B^\flat = A^\flat \cdot B_0[b_1, \ldots, b_n]$), and hence $f: B^\flat \to B^\flat$ is bijective (since $B^\flat$ is noetherian). Then $f(B^+) = B^+$, since B^+ is the integral closure of $A^+ \cdot B_0$ in $B^\flat$; and $f: B^\flat \to B^\flat$ is a homeomorphism, since the topology of B_0 is the $I \cdot B_0$-adic topology. Hence f is an isomorphism of affinoid rings.

(4) Let A be an affinoid ring, $T_1, \ldots, T_n$ finite subsets of $A^\flat$ with $T_i \cdot A^\flat$ open in $A^\flat$ for $i = 1, \ldots, n$, and let U be a rational subset of the affinoid adic space $X := \mathrm{Spa}A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}$. We consider X_i as an element of $\mathcal{O}_X(X)$ and hence we have $X_i|U \in \mathcal{O}_X(U)$ ($i = 1, \ldots, n$). There exists a neighbourhood H of 0 in $\mathcal{O}_X(U)$ such that every $\mathrm{Spa}A$-morphism $f: U \to X$ with $f^*(X_i) \in (X_i|U) + H$ for $i = 1, \ldots, n$ satisfies: $f(U) \subseteq U$ and $f: U \to U$ is an isomorphism of adic spaces. More precisely, for every neighbourhood G of 0 in $\mathcal{O}_X(U)$, there exists a neighbourhood H of 0 in $\mathcal{O}_X(U)$ such that every $\mathrm{Spa}A$-morphism $f: U \to X$ with $f^*(X_i) \in (X_i|U) + H$ for $i = 1, \ldots, n$ satisfies: $f(U) \subseteq U$ and $f: U \to U$ is an isomorphism with $(f^{-1})^*(X_i|U) \in (X_i|U) + G$ for $i = 1, \ldots, n$.

Proof. Let a neighbourhood G of 0 in $\mathcal{O}_X(U)$ be given. Let $s_0, s_1, \ldots, s_m$ be elements of $\mathcal{O}_X(X)$ such that $\{s_1, \ldots, s_m\} \cdot \mathcal{O}_X(X)$ is open in $\mathcal{O}_X(X)$ and $U = R(\frac{s_1, \ldots, s_m}{s_0})$. Then $\{s_1|U, \ldots, s_m|U\} \cdot \mathcal{O}_X(U)$ is open in $\mathcal{O}_X(U)$ and $U = \{u \in U \mid v_u(s_i|U) \leq v_u(s_0|U) \neq 0 \text{ for } i = 1, \ldots, m\}$. Then by [Hu$_1$, 3.10] there exists a neighbourhood H''' of 0 in $\mathcal{O}_X(U)$ such that, for $s_i' \in (s_i|U) + H'''(i = 1, \ldots, m)$, $U = \{u \in U \mid v_u(s_i') \leq v_u(s_0') \neq 0 \text{ for } i = 1, \ldots, m\}$. Let H'' be a neighbourhood of 0 in $\mathcal{O}_X(U)$ such that $f^*(s_i) \in (s_i|U) + H'''(i = 0, \ldots, m)$ for any Spa$A$-morphism $f: U \to X$ with $f^*(X_i) \in (X_i|U) + H''(i = 1, \ldots, n)$. Then $f(U) \subseteq U$ for any Spa$A$-morphism $f: U \to X$ with $f^*(X_i) \in (X_i|U) + H''$ for $i = 1, \ldots, n$. Let Q be a finite subset of $A^\flat$ such that $Q \cdot A^\flat$ is open in $A^\flat$ and $Q \cdot (s_0|U)^{-1} \in \mathcal{O}_X^+(U)$. Then we have the quotient mapping $q: (\mathcal{O}_X(X), \mathcal{O}_X^+(X))\langle X_{n+1}, \ldots, X_{n+m+1}\rangle_{\{1\}, \ldots, \{1\}, Q} \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ over $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ with $q(X_{n+i}) = \frac{s_i|U}{s_0|U}$ for $i = 1, \ldots, m$ and $q(X_{n+m+1}) = \frac{1}{s_0|U}$. Hence we have the quotient mapping $p: A\langle X_1, \ldots, X_{n+m+1}\rangle_{T_1, \ldots, T_n, \{1\}, \ldots, \{1\}, Q} \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ over A with $p(X_i) = X_i|U$ for $i = 1, \ldots, n$ and $p(X_i) = q(X_i)$ for $i = n+1, \ldots, n+m+1$. This shows that $X_1|U, \ldots, X_n|U, \frac{s_1|U}{s_0|U}, \ldots, \frac{s_m|U}{s_0|U}, \frac{1}{s_0|U}$ form a set of topological generators of $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ over A. Then by (3) there exists a neighbourhood H' of 0 in $\mathcal{O}_X(U)$ such that any Spa$A$-morphism $f: U \to U$ with $f^*(X_i|U) \in (X_i|U) + H'$ for $i = 1, \ldots, n$, $f^*(\frac{s_i|U}{s_0|U}) \in \frac{s_i|U}{s_0|U} + H'$ for $i = 1, \ldots, m$ and $f^*(\frac{1}{s_0|U}) \in \frac{1}{s_0|U} + H'$ is an isomorphism with $(f^{-1})^*(X_i|U) \in (X_i|U) + G$ for $i = 1, \ldots, n$. Let H be a neighbourhood of 0 in $H' \cap H''$ such that any Spa$A$-morphism $f: U \to U$ with $f^*(X_i|U) \in (X_i|U) + H$ (for $i = 1, \ldots, n$) satisfies $f^*(\frac{s_i|U}{s_0|U}) \in \frac{s_i|U}{s_0|U} + H'$ for $i = 1, \ldots, m$ and $f^*(\frac{1}{s_0|U}) \in \frac{1}{s_0|U} + H'$. Then any Spa$A$-morphism $f: U \to X$ with $f^*(X_i) \in (X_i|U) + H$ for $i = 1, \ldots, n$ satisfies: $f(U) \subseteq U$ and $f: U \to U$ is an isomorphism with $(f^{-1})^*(X_i|U) \in (X_i|U) + G$ for $i = 1, \ldots, n$.

After this preparations we can prove that (ii) implies (iii). We use the notations introduced in (ii). Put $Z := \text{Spa}B$ and $\Delta := \det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n} \in B^\flat = \mathcal{O}_Z(Z)$. Since $\Delta(x) \neq 0$ for any $x \in \text{Spa}B/I$, there exist $s_1, \ldots, s_m \in A^\flat$ such that $\{s_1, \ldots, s_m\} \cdot A^\flat$ is open in $A^\flat$ and $v_x(s_i) \leq v_x(\Delta) \neq 0$ for any $x \in \text{Spa}B/I$, $i = 1, \ldots, m$ ([Hu$_1$, 3.11]). Let U denote the rational subset $\{x \in Z \mid v_x(s_i) \leq v_x(\Delta) \neq 0 \text{ for } i = 1, \ldots, m\}$ of Z. Let H be a neighbourhood of 0 in $\mathcal{O}_Z(U)$ such that $t(h + X_i|U) \in \mathcal{O}_Z^+(U)$ for every $h \in H$, $i \in \{1, \ldots, n\}$, $t \in T_i$. Then, for any $h = (h_1, \ldots, h_n) \in H^{\times n}$, we have a Y-morphism $f_h: U \to Z$ with $f_h^*(X_i) = h_i + X_i|U$ for $i = 1, \ldots, n$ ([Hu, 3.5 i]). Decreasing H, we may assume by (4) that $f_h(U) \subseteq U$ and $f_h: U \to U$ is an isomorphism for any $h \in H^{\times n}$. Since $\Delta|U$ is a unit of $\mathcal{O}_Z(U)$ there exists by (2) a neighbourhood V of 0 in $\mathcal{O}_Z(U)$ such that, for every $(v_1, \ldots, v_n) \in V^{\times n}$, there exists a $h \in H^{\times n}$ with $f_h^*(f_i) = v_i + f_i|U$ for $i = 1, \ldots, n$. Since $Z - U$ is quasi-compact and, for every $x \in Z - U$, there exists a $i \in \{1, \ldots, n\}$ with $f_i(x) \neq 0$, there exists a neighbourhood W of 0 in $B^\flat$ such that, for every $x \in Z - U$ and every $(w_1, \ldots, w_n) \in W^{\times n}$, there exists a $i \in \{1, \ldots, n\}$ with $(w_i + f_i)(x) \neq 0$. Let S be a neighbourhood of 0 in $B^\flat$ such that $S \subseteq W$ and $\{s|U \mid s \in S\} \subseteq V$. Then, for any $(s_1, \ldots, s_n) \in S^{\times n}$, if we put $I' := \{f_1 + s_1, \ldots, f_n + s_n\} \cdot B^\flat$ then SpaB/I'

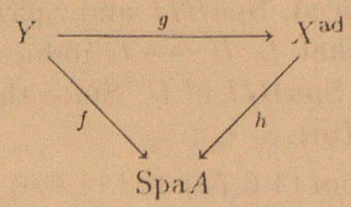
is Y-isomorphic to $\mathrm{Spa}B/I$ and the image of $\det(\frac{\partial(f_i+s_i)}{\partial X_j})_{i,j=1,\ldots,n} \in B^\triangleright$ in $B^\triangleright/I'$ is an unit of $B^\triangleright/I'$. Indeed, $\mathrm{Spa}B/I$ and $\mathrm{Spa}B/I'$ are contained in U, and there exists a $h \in H^{\times n}$ such that $f_h: U \xrightarrow{\sim} U$ induces an isomorphism between the adic subspaces $\mathrm{Spa}B/I'$ and $\mathrm{Spa}B/I$ of U. Since the image of $A^\triangleright[X_1, \ldots, X_n]$ in $B^\triangleright$ is dense in $B^\triangleright$, we obtain (iii).

(iii) $\Longrightarrow$ (i) follows from (1.6.7.i) and (1.6.9).

Corollary 1.7.2. i) *Let A be an affinoid ring and $T_1, \ldots, T_n$ finite subsets of $A^\triangleright$ such that $T_i \cdot A^\triangleright$ is open in $A^\triangleright$ for $i = 1, \ldots, n$. Let B be the affinoid ring $A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}$ and let $f_1, \ldots, f_n$ be elements of $B^\triangleright$ such that the image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n} \in B^\triangleright$ in $B^\triangleright/I$ is a unit of $B^\triangleright/I$ where $I := f_1B^\triangleright + \ldots + f_nB^\triangleright$. Let V be a neighbourhood of 0 in $B^\triangleright/I$. Then there exists a neighbourhood S of 0 in $B^\triangleright$ such that if $f_1' \in f_1 + S, \ldots, f_n' + S$ and $I' := f_1'B^\triangleright + \ldots + f_n'B^\triangleright$ then the image of $\det(\frac{\partial f_i'}{\partial X_j})_{i,j=1,\ldots,n} \in B^\triangleright$ in $B^\triangleright/I'$ is a unit of $B^\triangleright/I'$ and there exists a $\mathrm{Spa}A$-isomorphism $c: \mathrm{Spa}B/I \to \mathrm{Spa}B/I'$ with $c^*(\overline{T}_i') \in \overline{T}_i + V$ for $i = 1, \ldots, n$ where $\overline{T}_i \in B^\triangleright/I$ (resp. $\overline{T}_i' \in B^\triangleright/I'$) is the image of $T_i \in B^\triangleright$ in $B^\triangleright/I$ (resp. $B^\triangleright/I'$).*
ii) *Let $X \to Y = \mathrm{Spa}A$ be an étale morphism of affinoid adic spaces. If A is Tate then we have a description of X as in (1.7.1.iii) with $T_1 = \ldots = T_n = \{1\}$.*

Proof. i) We use the notations from the end of the proof of (1.7.1). We consider $\mathrm{Spa}B/I$ as an adic subspace of U. Let G be a neighbourhood of 0 in $\mathcal{O}_Z(U)$ such that, for any $g \in G$, the restriction $\bar{g} \in B^\triangleright/I$ of g to $\mathrm{Spa}B/I$ is contained in V. Decreasing H, we may assume by (4) in the proof of (1.7.1) that, for any $h \in H^{\times n}$, $f_h(U) \subseteq U$ and $f_h: U \to U$ is an isomorphism with $(f_h^{-1})^*(X_i|U) \in (X_i|U) + G$ for $i = 1, \ldots, n$. Then the arguments at the end of the proof of (1.7.1) show that the isomorphisms $f_h^{-1}: U \to U$ induce the isomorphism $c: \mathrm{Spa}B/I \to \mathrm{Spa}B/I'$ we are looking for.
ii) If an affinoid ring A is Tate then every affinoid ring of topologically finite type over A is a quotient of some $A\langle X_1, \ldots, X_n\rangle$ ([Hu, 3.5.iii]). Therefore ii) follows from i) and the proof of (1.7.1.i) $\Rightarrow$ (1.7.1.ii). □

Corollary 1.7.3. *Let A be an affinoid ring, D a ring and $\varphi: D \to A^\triangleright$ a ring homomorphism. φ induces a morphism of locally ringed spaces $\mathrm{Spa}A \to \mathrm{Spec}D$. Then by [Hu, 3.8] every scheme X locally of finite type over $\mathrm{Spec}D$ defines an adic space $X^{\mathrm{ad}} := X \times_{\mathrm{Spec}D} \mathrm{Spa}A$ locally of finite type over $\mathrm{Spa}A$.*
i) *If $X \to \mathrm{Spec}D$ is an étale morphism of schemes then $X^{\mathrm{ad}} \to \mathrm{Spa}A$ is an étale morphism of adic spaces.*
ii) *Let Y be an affinoid adic space and $f: Y \to \mathrm{Spa}A$ an étale morphism. If $\mathrm{im}(\varphi)$ is dense in $A^\triangleright$ then there exist an affine scheme X, an étale morphism $X \to \mathrm{Spec}D$ and an open embedding of adic spaces $g: Y \to X^{\mathrm{ad}}$ such that the diagram*



commutes where h is the structure morphism of X^{ad}.

iii) *Let Y be an affinoid adic space and $f: Y \to \mathrm{Spa}A$ a morphism of adic spaces. Then equivalent are*

a) *f is étale.*

b) *There exist an affinoid ring B and a ring homomorphism of algebraically finite type $A \to B$ between affinoid rings ((1.2.5)) such that $A^\triangleright \to B^\triangleright$ is étale (in the algebraic sense) and Y is $\mathrm{Spa}A$-isomorphic to $\mathrm{Spa}B$.*

Proof. i) We may assume that X is affine. Then there exist $f_1, \ldots, f_n \in A^\triangleright[X_1, \ldots, X_n] =: C$ such that X is $\mathrm{Spec}A^\triangleright$-isomorphic to $\mathrm{Spec}C/(f_1, \ldots, f_n)$ and the image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n}$ in $C/(f_1, \ldots, f_n)$ is an unit of $C/(f_1, \ldots, f_n)$. Let $x \in X^{ad}$ be given. Then there exist an open neighbourhood U of x in X^{ad} and finite subsets $T_1, \ldots, T_n$ of $A^\triangleright$ with $T_i \cdot A^\triangleright$ open in $A^\triangleright$ for $i = 1, \ldots, n$ such that U is $\mathrm{Spa}A$-isomorphic to $\mathrm{Spa}B/I$ with $B = A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}$ and $I = f_1 B^\triangleright + \ldots + f_n B^\triangleright$. Hence by (1.7.1) U is étale over $\mathrm{Spa}A$.

ii) By (1.7.1) and (1.7.2.i) there exist finite subsets $T_1, \ldots, T_n$ of $A^\triangleright$ with $T_i \cdot A^\triangleright$ open in $A^\triangleright$ for $i = 1, \ldots, n$ and $f_1, \ldots, f_n \in D[X_1, \ldots, X_n]$ such that Y is $\mathrm{Spa}A$-isomorphic to $\mathrm{Spa}B/I$ with $B := A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}$, $I := f_1 B^\triangleright + \ldots + f_n B^\triangleright$, and the image of $\det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n}$ in $B^\triangleright/I$ is an unit of $B^\triangleright/I$. Let Z be the $\mathrm{Spec}D$-scheme $\mathrm{Spec}D[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$. Then Y can be considered as an open subspace of Z^{ad}. Put $\Delta := \det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n}$ and $X := \{z \in Z \mid \Delta \text{ is an unit in } \mathcal{O}_{Z,z}\}$. Then X is an open affine subscheme of Z which is etale over $\mathrm{Spec}D$, and Y is an open subspace of X^{ad}.

iii) It is easily seen that if b) is satisfied then f has a factorization as in ii) and hence f is étale. Now we assume a) and show b). By (1.7.1) there exist finite subsets $T_1, \ldots, T_n$ of $A^\triangleright$ with $T_i \cdot A^\triangleright$ open in $A^\triangleright$ for $i = 1, \ldots, n$ and $f_1, \ldots, f_n \in A^\triangleright[X_1, \ldots, X_n]$ such that Y is $\mathrm{Spa}A$-isomorphic to $\mathrm{Spa}C$ with $C := A\langle X_1, \ldots, X_n\rangle_{T_1,\ldots,T_n}/(f_1, \ldots, f_n)$ and the image c of $d := \det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n}$ in $C^\triangleright$ is a unit of $C^\triangleright$. Since $c(x) \neq 0$ for every $x \in \mathrm{Spa}C$, there exists a finite subset T of $A^\triangleright$ such that $T \cdot A^\triangleright$ is open in $A^\triangleright$ (and then also $T \cdot C^\triangleright$ is open in $C^\triangleright$) and $|m(x)| \leq |c(x)| \neq 0$ for every $x \in \mathrm{Spa}C$, $m \in T$. Then $C = C\langle\frac{T}{c}\rangle = C\langle X\rangle_T/(1 - cX) = A\langle X_1, \ldots, X_n, X\rangle_{T_1,\ldots,T_n,T}/(f_1, \ldots, f_n, 1 - dX)$. Obviously there exists a ring homomorphism of algebraically finite type $A \to B$ between affinoid rings such that $A^\triangleright \to B^\triangleright$ is étale and C is A-isomorphic to the completion of B. □

(1.6.8) and (1.7.3.iii) imply

Corollary 1.7.4. *Every unramified morphism of adic spaces is locally quasi-finite.*

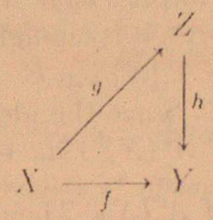
Proposition 1.7.5. *Let $f: X \to Y$ be a morphism of analytic adic spaces which is locally of finite type. Equivalent are*

a) *f is étale.*

b) *f is flat (i.e., for every $x \in X$, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$) and unramified.*

c) *For every maximal point x of X, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is a finite separable extension field of $k(f(x))$.*

Proof. a) $\Rightarrow$ b) by the subsequent Lemma (1.7.6).
b) $\Rightarrow$ a): We may assume that X and Y are affinoid. Then by (1.6.8) there exists a commutative diagram



where g is a closed embedding and h is étale. In order to show that f is étale it suffices to show that g is an open embedding. For that it suffices to show that g is flat.
Let $s: X \to X \times_Y Z$ be the graph morphism of g and $p: X \times_Y Z \to Z$ the projection. Since $g = p \circ s$, it suffices to show that p and s are flat. By (1.7.4) f is quasi-finite. Hence by (1.6.4.ii) p is flat. s is an open embedding, since s is the pull back of the diagonal morphism $\Delta: Z \to Z \times_Y Z$ by $g \times \mathrm{id}: X \times_Y Z \to Z \times_Y Z$ and Δ is an open embedding by (1.6.8).
c) $\Rightarrow$ b): f is flat by (1.1.10.iii). By (1.5.2) f is locally quasi-finite. Then by (1.6.4.i) and Nakayama's Lemma, $(\Omega_{X|Y})_x = 0$ for every maximal point x of X. Then $\Omega_{X|Y} = 0$, and hence by (1.6.8) f is unramified.
b) $\Rightarrow$ c): By (1.7.4) f is quasi-finite. Then, for every maximal point x of X and $A := \mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$, A is finite over $k(f(x))$ and $\Omega_{A|k(f(x))} = 0$ ((1.6.8) and (1.6.4)). Hence A is a finite separable extension field of $k(f(x))$. □

Lemma 1.7.6. *Let $X \to Y$ be an étale morphism between affinoid adic spaces. Then the ring homomorphism $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is flat.*

Proof. Let A be the affinoid ring $(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$. By (1.7.1) we may assume that there exist finite subsets $T_1, \ldots, T_n$ of $A^\flat$ with $T_i \cdot A^\flat$ open in $A^\flat$ for $i = 1, \ldots, n$ and $f_1, \ldots, f_n \in A^\flat[X_1, \ldots, X_n]$ such that if we put $B = A^\flat\langle X_1, \ldots, X_n\rangle_{T_1, \ldots, T_n}$ and $I = f_1 B + \ldots + f_n B$ then the image of $\Delta := \det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\ldots,n} \in A^\flat[X_1, \ldots, X_n]$ in B/I is a unit of B/I and $\mathcal{O}_X(X) = B/I$.
We distinguish two cases.
i) Assume that $A^\flat$ has a noetherian ring of definition.

Then there exists a f-adic topology on $A^\flat[X_1, \ldots, X_n]$ such that B is the completion of $A^\flat[X_1, \ldots, X_n]$ and $A^\flat[X_1, \ldots, X_n]$ has a noetherian ring of definition. Therefore B is flat over $A^\flat[X_1, \ldots, X_n]$ ([Hu$_1$, 1.7.i]), and hence B/I is flat over $C := A^\flat[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$. Let C_Δ be the localization of C by Δ. Then the ring homomorphism $C \to B/I$ factors through a ring homomorphism $C_\Delta \to B/I$ which is also flat. Since C_Δ is étale over $A^\flat$ and therefore flat over $A^\flat$, we obtain that B/I is flat over $A^\flat$.

ii) Assume that $A^\flat$ is a strongly noetherian Tate ring.

Then by (1.7.2.ii) we may assume $T_1 = \ldots = T_n = \{1\}$. We show that $A^\flat\langle X_1, \ldots, X_n\rangle$ is flat over $A^\flat[X_1, \ldots, X_n]$. (Then we can proceed as in (i)). Using induction on n, we may assume $n = 1$. Every $A^\flat$-module homomorphism $M \to N$ between finitely generated $A^\flat$-modules is strict if we equip M and N with its natural $A^\flat$-module topologies ([Hu, 2.4]). This implies that $A^\flat\langle X\rangle$ is flat over $A^\flat$, cf. [FP, III.7.8, 7.9]. Since also $A^\flat[X]$ is flat over $A^\flat$, it is enough to prove that, for every residue field k of $A^\flat$, the ring homomorphism $A^\flat[X] \otimes_{A^\flat} k \to A^\flat\langle X\rangle \otimes_{A^\flat} k$ is flat ([EGA, 0_{III}.10.2.5]). Let $\mathfrak{p}$ be a prime ideal of $A^\flat$. Since the ideal $\mathfrak{p} \cdot A^\flat\langle X\rangle$ is closed in $A^\flat\langle X\rangle$, we have $\mathfrak{p} \cdot A^\flat\langle X\rangle = \{\sum_{n \in \mathbb{N}_0} a_n X^n \in A^\flat\langle X\rangle \mid a_n \in \mathfrak{p} \text{ for every } n \in \mathbb{N}_0\}$. This shows that $\mathfrak{p} \cdot A^\flat\langle X\rangle$ is a prime ideal of $A^\flat\langle X\rangle$. Then $M := A^\flat\langle X\rangle/\mathfrak{p} \cdot A^\flat\langle X\rangle$ is a torsion-free C-module with $C := A^\flat[X]/\mathfrak{p} \cdot A^\flat[X]$. Let S be the multiplicative system $A^\flat - \mathfrak{p}$. Then the localization M_S is a torsion-free C_S-module. Since C_S is a Dedekind ring, we obtain that M_S is flat over C_S. □

(1.7.7) Let A be an affinoid ring and $f: Y \to \mathrm{Spa}A$ a smooth morphism of adic spaces. Then, for every $y \in Y$, there exist an open affinoid neighbourhood U of y in Y, an affinoid ring B and a ring homomorphism of algebraically finite type $A \to B$ between affinoid rings such that U is $\mathrm{Spa}A$-isomorphic to $\mathrm{Spa}B$ and the morphism of schemes $\mathrm{Spec}B^\flat \to \mathrm{Spec}A^\flat$ is smooth ((1.6.10), (1.7.3.iii)).

Proposition 1.7.8. *Every smooth morphism of adic spaces is open.*

Proof. The assertion follows from (1.7.7) and the subsequent Lemma (1.7.9). □

Lemma 1.7.9. *Let $f: A \to B$ be an adic morphism of affinoid rings such that the ring homomorphism $f: A^\flat \to B^\flat$ is flat and finitely presented and there exists a finite subset L of B^+ such that B^+ is the integral closure of $A^+[L]$ in $B^\flat$. Then $\mathrm{Spa}(f): \mathrm{Spa}B \to \mathrm{Spa}A$ is an open mapping.*

Proof. Put $\varphi := \mathrm{Spa}(f)$. Let $U = R(\frac{T}{s})$ be a rational subset of $\mathrm{Spa}B$. We have to show that $\varphi(U)$ is open in $\mathrm{Spa}A$. Replacing B by the affinoid ring $B(\frac{T}{s})$ ([Hu, §1]), we may assume $U = \mathrm{Spa}B$.

Put $X_A := \{v \in \mathrm{Spv}A^\flat \mid v(a) \leq 1 \text{ for every } a \in A^+ \text{ and } v(a) < 1 \text{ for every } a \in (A^\flat)^{\circ\circ}\} \subset \mathrm{Spv}A^\flat$ and $X_B := \{v \in \mathrm{Spv}B^\flat \mid v(b) \leq 1 \text{ for every } b \in A^+[L] \text{ and } v(b) < 1 \text{ for every } b \in (A^\flat)^{\circ\circ}\} = \{v \in \mathrm{Spv}B^\flat \mid v(b) \leq 1 \text{ for every } b \in B^+ \text{ and } v(b) < 1 \text{ for every } b \in (B^\flat)^{\circ\circ}\} \subset \mathrm{Spv}B$, and let $\psi: X_B \to X_A$ be the restriction of $\mathrm{Spv}(f): \mathrm{Spv}B^\flat \to \mathrm{Spv}A^\flat$. By [Hu$_1$, 2.6.iii] we have retractions $r_A: \mathrm{Spv}A \to$

(o) *The map*

$$|X| \to \varprojlim_i |X_i|$$

is a homeomorphism of spectral spaces.

(i) *The base change functors* $(X_i)_{\text{fét}} \to X_{\text{fét}}$ *induce an equivalence of categories*

$$2\text{-}\varinjlim_i (X_i)_{\text{fét}} \to X_{\text{fét}} .$$

(ii) *Let* $X_{\text{ét,qcqs}} \subset X_{\text{ét}}$ *denote the full subcategory of qcqs étale perfectoid spaces over* X *(and similarly for* X_i*). Then the base change functors* $(X_i)_{\text{ét,qcqs}} \to X_{\text{ét,qcqs}}$ *induce an equivalence of categories*

$$2\text{-}\varinjlim_i (X_i)_{\text{ét,qcqs}} \to X_{\text{ét,qcqs}} .$$

(iii) *Let* $X_{\text{ét,qc,sep}} \subset X_{\text{ét,qcqs}}$ *be the full subcategory of quasicompact separated étale perfectoid spaces over* X *(and similarly for* X_i*). Then the base change functors* $(X_i)_{\text{ét,qc,sep}} \to X_{\text{ét,qc,sep}}$ *induce an equivalence of categories*

$$2\text{-}\varinjlim_i (X_i)_{\text{ét,qc,sep}} \to X_{\text{ét,qc,sep}} .$$

(iv) *Let* $X_{\text{ét,aff}} \subset X_{\text{ét,qc,sep}}$ *be the full subcategory of affinoid perfectoid spaces which are étale over* X *(and similarly for* X_i*). Then the base change functors* $(X_i)_{\text{ét,aff}} \to X_{\text{ét,aff}}$ *induce an equivalence of categories*

$$2\text{-}\varinjlim_i (X_i)_{\text{ét,aff}} \to X_{\text{ét,aff}} .$$

Proof. For the set-theoretic part, we can by Lemma 4.1 assume that $\kappa' = \kappa_\lambda$ has cofinality larger than λ, in which case the cardinality of the direct limit of the R_i^+ is still less than κ_λ, and so R is κ-small.

Part (o) follows from [Hub93a, Proposition 3.9], cf. [Sch12, Lemma 6.13 (ii)]. The statement in part (i) is equivalent to

$$2\text{-}\varinjlim_i (R_i)_{\text{fét}} \to R_{\text{fét}}$$

being an equivalence of categories. This is a consequence [Sch12, Lemma 7.5 (i)] of a theorem of Elkik, [Elk73], (in the noetherian case) and Gabber–Ramero, [GR03, Proposition 5.4.53], in general.

For part (ii), we first check fully faithfulness. Thus, let $Y_i, Y_i' \to X_i$ be two qcqs étale perfectoid spaces over X_i, and denote by $Y_j, Y_j' \to X_j$ their pullbacks to X_j for $j \geq i$, and $Y, Y' \to X$ their pullbacks to X. Assume that two morphisms $f_i, g_i : Y_i \to Y_i'$ become equal after pullback to X, $f = g : Y \to Y'$. Let $\Gamma_{f_i}, \Gamma_{g_i} : Y_i \hookrightarrow Y_i \times_X Y_i' =: Z_i$ be the graphs of f_i and g_i. Then Γ_{f_i} and Γ_{g_i} are injections, and thus are determined by the quasicompact open image of $|Y_i|$ in $|Z_i|$. We know that after pullback along $|Z| \to |Z_i|$, $Z = Z_i \times_{X_i} X$, the two quasicompact open images $\Gamma_{f_i}(|Y_i|)$, $\Gamma_{g_i}(|Y_i|)$ agree. As $|Z| = \varprojlim_j |Z_j|$ is an inverse limit of spectral spaces along spectral maps, these two quasicompact open images agree after pullback to $|Z_j|$ for large j, which means that $\Gamma_{f_j} = \Gamma_{g_j}$, as desired. This proves faithfulness.

As an intermediate step to fullness, we check that if $f_i : Y_i \to Y_i'$ is a morphism over X whose pullback $f : Y \to Y'$ to X is an isomorphism, then the pullback $f_j : Y_j \to Y_j'$ to X_j is an isomorphism for large enough j. Note that the image of f_i is a quasicompact open subset, over which $Y' \to Y_i'$ factors; a standard quasicompactness argument implies that $Y_j' \to Y_i'$ factors over this open subset for large enough j, so that f_j is surjective for large enough j. Applying the same

reasoning to $\Delta_{f_i} : Y_i \to Y_i \times_{Y'_i} Y_i$ shows that Δ_{f_j} is surjective for large enough j. As Δ_{f_j} is always an injection, this implies that Δ_{f_j} is an isomorphism, which then implies that f_j is an isomorphism.

Now, let $f : Y \to Y'$ be a morphism over X; we need to see that this is a base change of a morphism $f_j : Y_j \to Y'_j$ over X_j for large enough j. The graph $\Gamma_f : Y \hookrightarrow Y \times_X Y'$ is a quasicompact open embedding. By approximation of quasicompact open subsets, we can find a quasicompact open immersion $V_j \subset Y_j \times_{X_j} Y'_j$ whose pullback agrees with the image of Γ_f. In particular, the projection $V_j \to Y_j$ of qcqs étale perfectoid spaces over X_j becomes an isomorphism after pullback to X. By the intermediate step, we see that $V_j \to Y_j$ becomes an isomorphism for large enough j; composing the inverse with the map $V_j \hookrightarrow Y_j \times_{X_j} Y'_j \to Y'_j$ gives the desired map $Y_j \to Y'_j$.

Finally, for essential surjectivity, take any qcqs étale $f : Y \to X$. It is enough to cover Y be quasicompact open subsets which descend to X_i for large i, as by fully faithfulness (and the assumption that Y is qcqs), the pieces will automatically glue for large enough i. Thus, we can assume that f is a composite of a rational open subset, a finite étale map, and a rational open subset. Each of these maps descends to X_i for large enough i (using (i)), finishing the proof of (ii).

For (iii), fully faithfulness follows from (ii). It remains to see that if $f_i : Y_i \to X_i$ is a qcqs étale map whose base change $f : Y \to X$ to X is separated, then the base change $f_j : Y_j \to X_j$ to X_j is separated for large enough j. The map $f_j : Y_j \to X_j$ is separated if and only if the quasicompact open immersion $\Delta_{f_j} : Y_j \to Y_j \times_{X_j} Y_j$ is also a closed immersion. But in general, if S_i, $i \in I$, is a cofiltered inverse system of spectral spaces (with spectral maps) and inverse limit S, then if $U_i \subset S_i$ is a quasicompact open subset whose inverse image $U \subset S$ is open and closed, then the inverse image $U_j \subset S_j$ is open and closed for large j. Indeed, the open and closed decomposition given by U descends to S_j for large enough j, and for large enough j, one of the two quasicompact open subsets in this decomposition is equal to U_j. Applied with $S_i = |Y_i \times_{X_i} Y_i|$, $S = |Y \times_X Y|$, we see that if Δ_f is an open and closed immersion, then Δ_{f_j} is an open and closed immersion for large enough j.

In (iv), fully faithfulness again follows from (ii). Essential surjectivity follows from [Hub96, Proposition 1.7.1], which holds for affinoid perfectoid spaces by using the theory of pseudocoherent sheaves of Kedlaya–Liu, [KL16]. □

7. Totally disconnected spaces

In this section, we introduce a notion of (strictly) totally disconnected spaces. These are analogues of profinite sets in our setting.

Definition 7.1. *A perfectoid space X is totally disconnected if X is qcqs and every open cover of X splits, i.e. if $\{U_i \subset X\}$ is an open cover of X, then $\bigsqcup U_i \to X$ has a splitting.*

The following characterization of spectral spaces for which every open cover splits is due to L. Fargues:

Lemma 7.2. *Let X be a spectral space. The following conditions are equivalent.*

(i) *Every open cover of X splits.*

(ii) *Every connected component of X has a unique closed point.*

(iii) *The functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ on sheaves on X is exact, i.e. commutes with all finite colimits.*

(iv) *The functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ on sheaves of abelian groups on X is exact.*

(v) *For all sheaves of abelian groups $\mathcal{F}$ on X, one has $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

(vi) *For all sheaves of abelian groups $\mathcal{F}$ on X, one has $H^1(X, \mathcal{F}) = 0$.*

Proof. For (i) implies (ii), note that the condition that any open cover splits passes to closed subsets (by adding the open complement to the open cover). Thus, we may assume that X is connected. But then if $x, y \in X$ are two distinct closed points, then $(X \setminus \{x\}) \sqcup (X \setminus \{y\}) \to X$ does not have a splitting. For the converse, let $\{U_i \subset X\}$ be an open cover of X. Let $c \in \pi_0 X$, corresponding to a connected component $X_c \subset X$. Take some U_i which contains the unique closed point of X_c; then $X_c \subset U_i$. It follows that there is some open and closed neighborhood $U_c \subset X$ such that $U_c \subset U_i$. Thus, we can find a cover of $\pi_0 X$ by open and closed subsets such that $\coprod U_i \to X$ splits on the preimage. As $\pi_0 X$ is profinite, we can find a disjoint cover of $\pi_0 X$ by such subsets, and then assemble the local splittings into a splitting of $\coprod U_i \to X$.

Moreover, (i) implies (iii): We have to see that $\Gamma(X, -)$ commutes with coequalizers. Let $\mathcal{H} = \mathrm{coeq}(\mathcal{F} \rightrightarrows \mathcal{G})$ be a coequalizer diagram. First, we check that

$$\mathrm{coeq}(\Gamma(X, \mathcal{F}) \rightrightarrows \Gamma(X, \mathcal{G})) \to \Gamma(X, \mathcal{H})$$

is injective. If $t, t' \in \Gamma(X, \mathcal{G})$ are sections whose images in $\Gamma(X, \mathcal{H})$ agree, then after a cover $\{U_i \to X\}$, one can find $s_i \in \mathcal{F}(U_i)$ mapping to $t|_{U_i}$ and $t'|_{U_i}$. But under (i), we can split $\coprod U_i \to X$, giving us such a section $s \in \Gamma(X, \mathcal{F})$. Now assume $s \in \Gamma(X, \mathcal{H})$ is a section. Then after a cover $\{U_i \to X\}$, we can lift it to sections $s_i \in \mathcal{G}(U_i)$. But under (i), we can split $\coprod U_i \to X$, giving us a lift of s to $\Gamma(X, \mathcal{G})$, proving the desired surjectivity.

It is clear that (iii)⇒(iv)⇒(v)⇒(vi)⇒(iv). To see that (iv) implies (i), let $\{U_1, \dots, U_n\}$ be a finite quasicompact open cover of X, let $j_i : U_i \to X$ be the open immersions, and consider the surjection

$$\mathcal{F} = \bigoplus_i j_{i!}\mathbb{Z} \to \mathbb{Z}$$

of sheaves on X. Under condition (iv), this is surjective on global sections. This implies that there are locally constant functions $f_i : X \to \mathbb{Z}$ with $V_i = \mathrm{supp}\, f_i \subset U_i$ such that $\sum_i f_i = 1$. In particular, all V_i are open and closed, and

$$X = \coprod_i (V_i \setminus \bigcup_{j<i} V_j) \to \coprod_i U_i \to X$$

gives a section, as desired. □

For perfectoid spaces, one can give a finer structural analysis.

Lemma 7.3. *Let X be a totally disconnected perfectoid space.*

(i) *There is a continuous projection $\pi : X \to \pi_0(X)$ to the profinite set $\pi_0(X)$ of connected components.*

(ii) *All fibres of π are of the form $\mathrm{Spa}(K, K^+)$ for some perfectoid field K with an open and bounded valuation subring $K^+ \subset K$.*

Conversely, if X is a qcqs perfectoid space all of whose connected components are of the form $\mathrm{Spa}(K, K^+)$ for some perfectoid field K with an open and bounded valuation subring $K^+ \subset K$, then X is totally disconnected.

Proof. Part (i) is a consequence of the assumption that the underlying topological space of X is a spectral space. For part (ii), note that in a spectral space, any connected component is an intersection of open and closed subsets; this implies that connected components of qcqs perfectoid spaces are again perfectoid spaces. Thus, for (ii), we have to classify the connected totally disconnected perfectoid spaces. By Lemma 7.2, X has a unique closed point. By Example 5.2, it follows that if $x \in X$ is the unique closed point of a perfectoid space X, then $X = \mathrm{Spa}(K(x), K(x)^+)$, as desired.

The converse follows from Lemma 7.2, as then all connected components have a unique closed point. □

In [BS15], we worked with a stronger notion of w-local spaces; this includes the extra condition that the subset of closed points is closed. This stronger notion will have little relevance for us.

Rem. 1.3.17 [Hub96]

Definition 7.4. *A perfectoid space X is w-local if its underlying topological space is a w-local spectral space in the sense of* [BS15, Definition 2.1.1]. *Equivalently, X is qcqs, for any open cover $X = \bigcup_{i \in I} U_i$, the map $\bigsqcup_{i \in I} \to X$ splits, and the subset $X^c \subset X$ of closed points is closed.*

In particular, w-local spaces are totally disconnected.

(1.4.2) p. 61 [Hub96]

Lemma 7.5. *Let X be a totally disconnected perfectoid space. Then X is affinoid.*

Proof. Let $c \in \pi_0(X)$ be any point, and let $x_c \in X$ be the unique closed point mapping to c. There is an open affinoid neighborhood $U \subset X$ of x_c; in particular, U contains the connected component $X_c \subset X$, as all points of X_c specialize to x_c. We claim that there is an open compact neighborhood V of c in $\pi_0(X)$ such that $\pi^{-1}(V) \subset U$. Indeed, the intersection

L. 1.4.3 p. 61 [Hub96]

$$\bigcap_{c \in V} (X \setminus U) \cap \pi^{-1}(V) = \emptyset\,,$$

where the intersection runs over open and closed subsets $V \subset \pi_0(X)$ containing c. By quasicompactness of the constructible topology, it follows that there is some V such that $(X \setminus U) \cap \pi^{-1}(V) = \emptyset$, i.e. $\pi^{-1}(V) \subset U$. In this case, U has a closed and open decomposition $U = \pi^{-1}(V) \sqcup \pi^{-1}(\pi_0(X) \setminus V)$, so that $\pi^{-1}(V)$ is also affinoid. Thus, $\pi_0(X)$ is covered by open and closed subsets $V \subset \pi_0(X)$ such that $\pi^{-1}(V) \subset X$ is affinoid. By quasicompactness, finitely many such V cover $\pi_0(X)$, and we may moreover assume that they form a disjoint cover $\pi_0(X) = V_1 \sqcup \ldots \sqcup V_n$. Then X is the disjoint union $\pi^{-1}(V_1) \sqcup \ldots \sqcup \pi^{-1}(V_n)$, and thus affinoid. □

Lemma 15.2 Etale Homology of schemes

Lemma 7.6. *Let X be a totally disconnected perfectoid space. Let $U \subset X$ be a pro-constructible generalizing subset. Then U is an intersection of subsets of the form $\{|f| \leq 1\}$ for varying $f \in H^0(X, \mathcal{O}_X)$. In particular, U is an affinoid perfectoid space, which is moreover totally disconnected.*

Remark 7.7. Note that the lemma implies in particular that any quasicompact open subset of X is affinoid.

Proof. Let $x \in X \setminus U$ be a point, lying in a connected component $c = \pi(x) \in \pi_0(X)$. As the fibre X_c is of the form $\mathrm{Spa}(K_c, K_c^+)$ for some perfectoid field K_c and open and bounded valuation subring $K_c^+ \subset K_c$, one can find some $f_c \in K_c$ such that $|f_c| \leq 1$ on $U_c = U \cap X_c$, but $|f_c(x)| > 1$. Modulo K_c^+, we can lift f_c to a function f_V on $\pi^{-1}(V) \subset X$ for some compact open neighborhood $V \subset \pi_0(X)$ of c. Then by assumption

$$\bigcap_{c \in V' \subset V} (U \cap \{|f_V| > 1\}) \cap \pi^{-1}(V') = \emptyset\,.$$

Here, all terms are pro-constructible subsets of X, and thus compact in the constructible topology; by Tychonoff, it follows that for some open compact neighborhood V' of v, $(U \cap \{|f_V| > 1\}) \cap \pi^{-1}(V') = \emptyset$; in other words, after replacing V by V', we can assume that $U \cap \pi^{-1}(V) \subset \{|f_V| \leq 1\}$. Extending f_V by 0 on $\pi^{-1}(\pi_0(X) \setminus V)$, we get a function $f \in H^0(X, \mathcal{O}_X)$ with $|f| \leq 1$ on U, but $|f(x)| > 1$. Taking the intersection over all such f for varying $x \in X \setminus U$ gives U.

Thus, U is affinoid. Every connected component of U is an intersection of a connected component $\mathrm{Spa}(K_c, K_c^+)$ of X with U, which is necessarily of the form $\mathrm{Spa}(K_c, (K_c^+)')$ for another open and bounded valuation subring $(K_c^+)' \subset K_c$. □

The notion of w-local and totally disconnected spaces is useful as on the hand, they are rather simple, but on the other hand, there are many examples. More precisely, any qcqs perfectoid space X admits a universal map $X^{\mathrm{wl}} \to X$ from a w-local space X^{wl}, and the map $X^{\mathrm{wl}} \to X$ is pro-étale. Before we formulate this result, we recall the definition of pro-étale morphisms. We recall that Convention 4.2 is in place throughout the whole text, but we stress it again in the following definition.

Definition 7.8. *Let $f : Y \to X$ be a map of κ-small perfectoid spaces.*

(i) *The map f is* affinoid pro-étale *if $Y = \mathrm{Spa}(S, S^+)$ and $X = \mathrm{Spa}(R, R^+)$ are affinoid, and one can write $Y = \varprojlim Y_i \to X$ as a cofiltered inverse limit of étale maps $Y_i \to X$ from affinoid perfectoid spaces $Y_i = \mathrm{Spa}(S_i, S_i^+)$ along a κ-small index category I. Any such presentation $Y = \varprojlim Y_i \to X$ is called a* pro-étale presentation.

(ii) *The map f is* pro-étale *if for all $y \in Y$, there is an open neighborhood $V \subset Y$ of y and an open subset $U \subset X$ such that $f(V) \subset U$, and the restriction $f|_V : V \to U$ is affinoid pro-étale.*

For example, if $X = \mathrm{Spa}(C, \mathcal{O}_C)$ is a geometric point, then any affinoid pro-étale map $Y \to X$ is of the form $Y = X \times \underline{S}$ for some profinite set S, where if $S = \varprojlim_i S_i$ is a limit of finite sets, we set

$$X \times \underline{S} = \varprojlim_i X \times S_i .$$

Remark 7.9. A slightly curious example of pro-étale morphisms are Zariski closed immersions: If $f : Z \hookrightarrow X$ is a Zariski closed immersion of affinoid perfectoid spaces, then f is affinoid pro-étale. Indeed, if $Z = \{\forall f \in I : f = 0\} \subset X$ for an ideal $I \subset R = \mathcal{O}_X(X)$, then Z can be written as the intersection of the rational subsets

$$U_{f_1,\dots,f_n} = \{|f_1|, \dots, |f_n| \leq 1\}$$

over varying n and $f_1, \dots, f_n \in I$. Thus, $Z = \varprojlim U_{f_1,\dots,f_n} \to X$, which gives a pro-étale presentation of Z.

In particular, if $f : Y \to X$ is any morphism of perfectoid spaces, then $\Delta_f : Y \to Y \times_X Y$ is pro-étale. Indeed, this can be checked locally, so we can assume that X and Y are affinoid; then Δ_f is a Zariski closed immersion, and thus affinoid pro-étale.

Proposition 7.10. *Fix an affinoid perfectoid space X, and let $X_{\text{ét}}^{\text{aff}}$ be the category of étale maps $f : Y \to X$ from affinoid perfectoid spaces Y, and $X_{\text{proét}}^{\text{aff}}$ the category of affinoid pro-étale maps $f : Y \to X$. Then the functor*

$$\mathrm{Pro}_\kappa(X_{\text{ét}}^{\text{aff}}) \to X_{\text{proét}}^{\text{aff}} \ : \ \text{“}\varprojlim_i\text{”}\, Y_i \mapsto \varprojlim_i Y_i$$

is an equivalence of categories.

Here, Pro_κ denotes the category of pro-systems whose index category is bounded by κ.

Proof. By definition, the functor is essentially surjective, so we need to see that it is fully faithful. For this, write $X = \mathrm{Spa}(R, R^+)$, and take $Y = \mathrm{Spa}(S, S^+)$, $Z = \mathrm{Spa}(T, T^+)$ two objects of $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$ that are written as limits of $Y_i = \mathrm{Spa}(S_i, S_i^+) \in X^{\mathrm{aff}}_{\mathrm{\acute{e}t}}$, $Z_j = \mathrm{Spa}(T_j, T_j^+) \in X^{\mathrm{aff}}_{\mathrm{\acute{e}t}}$, for cofiltered index categories I, J. We need to show that

$$\mathrm{Hom}_X(Y, Z) = \varprojlim_j \varinjlim_i \mathrm{Hom}_X(Y_i, Z_j).$$

Without loss of generality J is a singleton, so $Z \to X$ is étale. Now have to check that

$$\mathrm{Hom}_X(Y, Z) = \varinjlim_i \mathrm{Hom}_X(Y_i, Z).$$

Recall that $Y_{\mathrm{\acute{e}t,qcqs}} = 2\text{-}\varinjlim_i (Y_i)_{\mathrm{\acute{e}t,qcqs}}$ by Proposition 6.4 (ii). Thus,

$$\mathrm{Hom}_X(Y, Z) = \mathrm{Hom}_Y(Y, Y \times_X Z) = \varinjlim_i \mathrm{Hom}_{Y_i}(Y_i, Y_i \times_X Z) = \varinjlim_i \mathrm{Hom}_X(Y_i, Z),$$

as desired. □

We need some basic properties of pro-étale maps.

Lemma 7.11.
(i) *Let $g : Z \to Y$ and $f : Y \to X$ be pro-étale (resp. affinoid pro-étale) morphisms. Then $f \circ g : Z \to X$ is pro-étale (resp. affinoid pro-étale).*
(ii) *Let $f : Y \to X$ be pro-étale (resp. affinoid pro-étale), and let $g : X' \to X$ be a map from a perfectoid space (resp. affinoid perfectoid space). Then $f' : Y' = X' \times_X Y \to X'$ is pro-étale (resp. affinoid pro-étale).*
(iii) *Let $f : Y \to X$ and $f' : Y' \to X$ be pro-étale (resp. affinoid pro-étale). Then any map $g : Y \to Y'$ over X is pro-étale (resp. affinoid pro-étale).*
(iv) *For any affinoid perfectoid space X, the category $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$ has all κ-small limits.*

Proof. It is enough to prove the assertions in the affinoid pro-étale case. For part (i), we use Proposition 7.10, Proposition 6.4 (iv) and the natural functor $\mathrm{Pro}_\kappa(\mathrm{Pro}_\kappa(\mathcal{C})) \to \mathrm{Pro}_\kappa(\mathcal{C})$, which exists for any category $\mathcal{C}$. Part (ii) follows directly from the definition. For part (iii), we can factor g as the composite of a section of the affinoid pro-étale map $Y \times_X Y' \to Y$ (given by the graph), and the affinoid pro-étale map $Y \times_X Y' \to Y'$; thus, it is enough to show that sections of affinoid pro-étale maps are affinoid pro-étale. But if $Y = \varprojlim Y_i \to X$ is a pro-étale presentation, and $s : X \to Y$ is a section, then s is given by a compatible system of sections $s_i : X \to Y_i$, and $X = \varprojlim_i (X \times_{Y_i} Y) \to Y$ is a pro-étale presentation.

Finally, for part (iv), as $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}} = \mathrm{Pro}_\kappa(X^{\mathrm{aff}}_{\mathrm{\acute{e}t}})$ by Proposition 7.10, it is enough to prove that $X^{\mathrm{aff}}_{\mathrm{\acute{e}t}}$ has all finite limits. For this, it is enough to check that it has a final object and fibre products, which is clear. □

Proposition 7.12. *The inclusion from the category of w-local perfectoid spaces with w-local maps, to the category of qcqs perfectoid spaces, admits a left adjoint $X \mapsto X^{\mathrm{wl}}$. The adjunction map $X^{\mathrm{wl}} \to X$ is pro-étale. If X is affinoid, the map $X^{\mathrm{wl}} \to X$ is an inverse limit of surjective maps of the form $\bigsqcup_{i \in I} U_i \to X$, where I is a finite set, and $U_i \subset X$ are rational subsets; in particular, it is affinoid pro-étale.*

Proof of (1). Let X^c be the set X equipped with the topology generated by the constructible subsets of X. Then X^c is quasi-compact. Let p be the mapping $X_v^c \to X^c$, $(x, A) \mapsto x$. Then p is continuous. For every $x \in X^c$, $p^{-1}(x)$ is homeomorphic to the abstract Riemann surface of the residue class field of $k(x)^+$ equipped with the constructible topology. Hence $p^{-1}(x)$ is quasi-compact. It remains to show that p is closed.

Let $\mathcal{D}$ be the set of the subsets of X_v of the form $D(L; U, E, F) := \{(x, A) \in X_v \mid x \in L,\ c(x) \notin A$ for every $c \in E$, $f(x) \in A$ for every $f \in F\}$ where $L \subseteq X$ is constructible, $U \subseteq X$ is open and constructible with $L \subseteq U$, and $E, F \subseteq \mathcal{O}_X(U)$ are finite subsets. Let $\mathcal{E}$ be the set of subsets of X_v which are finite unions of elements of $\mathcal{D}$. Then $\mathcal{E}$ is a boolean algebra of closed subsets of X_v^c. Let S be a closed subset of X_v^c. Then there exists a subset $\mathcal{F}$ of $\mathcal{E}$ with $S = \bigcap_{F \in \mathcal{F}} F$. We may assume that $F_1 \cap F_2 \in \mathcal{F}$ for every $F_1, F_2 \in \mathcal{F}$. Since all fibres of p are quasi-compact, we have $p(S) = \bigcap_{F \in \mathcal{F}} p(F)$. Thus it suffices to show that every $p(F)$ is closed in X^c. Hence it suffices to show that, for every $D \in \mathcal{D}$, $p(D)$ is closed in X^c.

Let $D = D(L; U, E, F) \in \mathcal{D}$ be given with $E = \{c_1, \ldots, c_n\}$ and $F = \{f_1, \ldots, f_m\}$. Let $\mathcal{P}$ be the set of all polynomials $a(Y, X) = \sum_{\nu \in \mathbb{H}_0^n} a_\nu(Y) X^\nu \in \mathbb{Z}[Y][X] = \mathbb{Z}[Y_1, \ldots, Y_m][X_1, \ldots, X_n]$ such that there exists a $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{H}_0^n$ such that $a_\mu(Y) = 1$ and, for every $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{H}_0^n$ with $a_\nu(Y) \neq 0$, $\nu_i \leq \mu_i$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n \nu_i < \sum_{i=1}^n \mu_i$. The following point shows that $p(D)$ is closed in X^c.

(2) $p(D) = \{x \in L \mid |f_i(x)| \leq 1$ for $i = 1, \ldots, m$ and $|a(f_1, \ldots, f_m, c_1, \ldots, c_n)(x)| \geq 1$ for every $a(Y, X) \in \mathcal{P}\}$.

Proof of (2). Let $x \in p(D)$ be given. Then $x \in L$ and there exists a $(x, A) \in X_v$ with $f_i(x) \in A$ for $i = 1, \ldots, m$ and $c_i(x) \notin A$ for $i = 1, \ldots, n$. Since $A \subseteq k^+(x)$, we have $f_i(x) \in k(x)^+$, i.e. $|f_i(x)| \leq 1$ for $i = 1, \ldots, m$. For every $a(Y, X) \in \mathcal{P}$ we have $b := a(f_1(x), \ldots, f_m(x), c_1(x), \ldots, c_n(x)) \notin A$. Then b is not contained in the maximal ideal of $k^+(x)$, i.e. $|a(f_1, \ldots, f_m, c_1, \ldots, c_n)(x)| \geq 1$.

Conversely, let x be an element of L with $|f_i(x)| \leq 1$ for $i = 1, \ldots, m$ and $|a(f_1, \ldots, f_m, c_1, \ldots, c_n)(x)| \geq 1$ for every $a(Y, X) \in \mathcal{P}$. We show that $x \in p(D)$. We have $X_v \in \mathcal{P}$. Hence $c_i(x) \neq 0$ and $\frac{1}{c_i(x)} \in k(x)^+$. Let K be the residue class field of $k(x)^+$, let $\lambda : k(x)^+ \to K$ be the natural ring homomorphism, and let $C \subseteq K$ be the subring generated by $\{\lambda(f_i(x)) \mid i = 1, \ldots, m\} \cup \{\lambda(\frac{1}{c_i(x)}) \mid i = 1, \ldots, n\}$. Since $|a(f_1(x), \ldots, f_m(x), c_1(x), \ldots, c_n(x))| \geq 1$ for every $a(Y, X) \in \mathcal{P}$, we have $C \neq \sum_{i=1}^n \lambda(\frac{1}{c_i(x)}) \cdot C$ (notice that $\lambda(\frac{1}{c_i(x)}) = 0$ if $c_i(x) \notin k(x)^+$). Hence there exists a valuation ring B of K such that $C \subseteq B$ and $\{\lambda(\frac{1}{c_i(x)}) \mid i = 1, \ldots, n\}$ is contained

in the maximal ideal of B. Then $A := \lambda^{-1}(B)$ is a valuation ring of $k(x)$ with $(x, A) \in D$. □

Lemma 1.3.13. *Let $f: X \to Y$ be a partially proper morphism between analytic adic spaces with Y quasi-compact and quasi-separated. Then, for every quasi-compact subset T of X, the closure $\overline{T}$ of T in X is quasi-compact.*

Proof. We may assume that T is open in X and Y is affinoid. Put $S := \{(x, A) \in X_v \mid x \in T$ and (x, A) has a center on $Y\} \subset X_v$. By (1.3.9.i) and (1.3.6.ii) every $(x, A) \in S$ has a unique center on X. Let $c: S \to X$ be the map which assigns to $(x, A) \in S$ the center of (x, A) on X. We equip S with the subspace topology of X_v. Then

(1) S is quasi-compact and $c: S \to X$ is continuous.

Proof. S is a topological subspace of T_v. By (1.3.6.ii) we have $S = \{(x, A) \in T_v \mid f(x) \in A$ for every $f \in \mathcal{O}_Y^+(Y)\}$. By (1.3.12) T_v is a spectral space and S is a pro-constructible subset of T_v. Hence S is quasi-compact.
Let U be an open subset of X. We show that $c^{-1}(U)$ is open in S. Since f is locally of $^+$weakly finite type, we may assume that U is affinoid and there exists a finite subset E of $\mathcal{O}_X(U)$ such that $\mathcal{O}_X^+(U)$ is the smallest ring of integral elements of $\mathcal{O}_X(U)$ which contains E and the image of $\mathcal{O}_Y^+(Y)$ in $\mathcal{O}_X(U)$. Then by (1.3.6.ii), $c^{-1}(U) = \{(x, A) \in S \mid x \in U$ and $e(x) \in A$ for every $e \in E\}$. Hence $c^{-1}(U)$ is open in S.

By (1) $c(S) \subset X$ is quasi-compact. By (1.3.6.i) $c(S)$ is the set of the specializations of the points of T in X. Since T is pro-constructible in X, we have $T = c(S)$.

□

For later use we remark

Lemma 1.3.14. *Let S be an analytic adic space, X' and Y analytic adic S-spaces, X an open subspace of X' and $f: X \to Y$ a S-morphism.*
i) *If every point of X' is a specialization of a point of X and Y is separated over S then there exists at most one S-morphism $g: X' \to Y$ with $f = g|X$.*
ii) *Let $g: X' \to Y$ be an S-morphism with $f = g|X$. If every point of X' is a specialization of a point of X, X' is separated over S, X' is quasi-compact, Y is quasi-separated, f is injective and, for every $x \in X$, the residue class field $k(f(x))$ is dense in the residue class field $k(x)$ then g is a homeomorphism onto its image.*
iii) *We assume that Y is locally of $^+$weakly finite type over S, the inclusion $X \hookrightarrow X'$ is quasi-compact and, for every open subset V of X', the restriction mapping $\mathcal{O}_{X'}(V) \to \mathcal{O}_X(V \cap X)$ is an isomorphism of topological rings. Let x' be a point of X' and y a point of Y such that there exists a valuation ring (x, A) of X such that x' is a center of (x, A) on X', y is a center of $f_*((x, A))$ on Y, x' and y lie over the same point of S, and x is the closed point of the set of the generalizations of x' in X (cf. (1.1.10.i)). Then there exist an open neighbourhood U' of x' in X' and a S-morphism $g: U' \to Y$ with $g|U' \cap X = f|U' \cap X$ and $y = g(x')$.*

Proof. By [BS15, Lemma 2.1.10], there is a similar w-localization $|X|^{\mathrm{wl}}$ of the underlying topological space $|X|$. Consider the topological space $|X|^{\mathrm{wl}}$ endowed with the presheaves $\mathcal{O}^+_{X^{\mathrm{wl}}}$, $\mathcal{O}_{X^{\mathrm{wl}}}$ of complete topological rings given by $\mathcal{O}^+_{X^{\mathrm{wl}}} = (p^*\mathcal{O}^+_X)^\wedge_\varpi$, $\mathcal{O}_{X^{\mathrm{wl}}} = \mathcal{O}^+_{X^{\mathrm{wl}}}[\varpi^{-1}]$, where $p : |X|^{\mathrm{wl}} \to |X|$ denotes the projection, and ϖ is a (local choice of) pseudouniformizer. We claim that this defines a perfectoid space X^{wl}; it is then clear that it is final in the category of w-local perfectoid spaces over X (with w-local maps).

We handle the affinoid case first. We use the following general presentation of the w-localization.

Lemma 7.13. *Let X be a spectral topological space, and let B be a basis for the topology consisting of quasicompact open subsets, such that for $U, V \in B$, also $U \cap V \in B$. Let $X^{\mathrm{wl}} \to X$ be the w-localization of X, and let C be the category of factorizations $X^{\mathrm{wl}} \to T \to X$, where T is a finite disjoint union of elements of B. Then C is cofiltered, and the natural map*

$$f : X^{\mathrm{wl}} \to \varprojlim_C T$$

is a homeomorphism.

Proof. Note that the category of C is cofiltered: Given any two factorizations $X^{\mathrm{wl}} \to T_1 \to X$ and $X^{\mathrm{wl}} \to T_2 \to X$, one gets another one $X^{\mathrm{wl}} \to T_1 \times_X T_2 \to X$, and for any two given maps $f, g : T_1 \to T_2$ between two such factorizations, the locus $T \subset T_1$ where $f = g$ is open and closed (more precisely, if $T_1 = \bigsqcup_{i \in I} U_i$, then $T = \bigsqcup_{i \in J} U_i$ for some subset $J \subset I$), and $X^{\mathrm{wl}} \to T_1$ factors over T. In particular, we can assume that $X \in B$, as the two resulting categories are cofinal.

By Lemma 2.5, it is enough to see that $f : X^{\mathrm{wl}} \to \varprojlim_C T$ is bijective and generalizing. Recall that $\pi_0(X^{\mathrm{wl}}) = X^{\mathrm{cons}}$, where X^{cons} is X endowed with the constructible topology, and the connected component corresponding to $x \in X$ is given by the localization X_x, i.e. the set of points specializing to x. In particular, points of X^{wl} are in bijection with pairs (x, y) of points in X, where $y \in X_x$. The map $X^{\mathrm{wl}} \to X$ is given by $(x, y) \mapsto y$, which is generalizing; more precisely, for any $(x, y) \in X^{\mathrm{wl}}$, the localization $X^{\mathrm{wl}}_{(x,y)}$ is identified with the localization X_y. As $T \to X$ is a finite disjoint union of open subsets, it also identifies localizations, and thus the map $f : X^{\mathrm{wl}} \to \varprojlim_C T$ identifies localizations, and in particular is generalizing.

For injectivity of $f : X^{\mathrm{wl}} \to \varprojlim_C T$, it is enough to show that given points $(x, y), (x', y') \in X^{\mathrm{wl}}$ with $x \neq x'$, $f(x, y) \neq f(x', y')$ (as the y-coordinate is seen by the projection to X). For this, choose some $U \in B$ such that U contains exactly one of x and x', say x, and look at $T = U \sqcup X \to X$. We have a factorization

$$X^{\mathrm{wl}} = X^{\mathrm{wl}} \times_{X^{\mathrm{cons}}} U^{\mathrm{cons}} \sqcup X^{\mathrm{wl}} \times_{X^{\mathrm{cons}}} (X \setminus U)^{\mathrm{cons}} \to U \sqcup X = T \to X$$

which sends (x, y) into $U \subset T$, and (x', y') into $X \subset T$. Thus, we get injectivity.

For surjectivity, we claim that it is enough to show that if $X^{\mathrm{wl}} \to T \to X$ is a factorizaton, and $t \in T$ is a point that is not in the image of $X^{\mathrm{wl}} \to T$, then one can find another factorization $X^{\mathrm{wl}} \to \widetilde{T} \to T \to X$ such that t is not in the image of $\widetilde{T} \to T$. Indeed, this implies that the image of $X^{\mathrm{wl}} \to T$ agrees with the image $T_0 \subset T$ of $\varprojlim_C T \to T$. But then $\varprojlim_C T_0 = \varprojlim_C T$, and we have a family of surjective maps $X^{\mathrm{wl}} \to T_0$; by a usual quasicompactness argument, this implies that $X^{\mathrm{wl}} \to \varprojlim_C T_0 = \varprojlim_C T$ is surjective (the fibre over any point is a cofiltered limit of nonempty spectral spaces along spectral maps, thus nonempty by Tychonoff applied to the constructible topology).

Thus, assume $t \in T$ is not in the image of $X^{\mathrm{wl}} \to T = \bigsqcup_{i \in I} U_i$. Assume $t \in U_{i_0}$, and choose finitely many $V_j \subset U_{i_0}$, $j \in J$, $V_j \in B$ with $t \notin V_j$, such that $X^{\mathrm{wl}} \to T$ factors over

$$\bigcup_{j \in J} V_j \sqcup \bigsqcup_{i \in I, i \neq i_0} U_i .$$

This is possible, as the map $X^{\mathrm{wl}} \to T$ is generalizing, so if t is not in the image, the closure of t is not in the image as well. We have an open cover

$$\widetilde{T} := \bigsqcup_{j \in J} V_j \sqcup \bigsqcup_{i \in I, i \neq i_0} U_i \to \bigcup_{j \in J} V_j \sqcup \bigsqcup_{i \in I, i \neq i_0} U_i .$$

After pullback to X^{wl}, this open cover splits by the definition of w-local spaces. In particular, the map

$$X^{\mathrm{wl}} \to \bigcup_{j \in J} V_j \sqcup \bigsqcup_{i \in I, i \neq i_0} U_i$$

lifts to a map $X^{\mathrm{wl}} \to \widetilde{T}$, giving the desired factorization $X^{\mathrm{wl}} \to \widetilde{T} \to T \to X$, with t not in the image of $\widetilde{T} \to T$. □

We apply this in the case where B is the basis of rational subsets of X. Clearly, any T as in the lemma admits a canonical structure as an affinoid perfectoid space, for which $T \to X$ is étale. Then $\varprojlim_C T = X^{\mathrm{wl}}$, on topological spaces by Lemma 7.13, and on $\mathcal{O}$ and $\mathcal{O}^+$ by a direct verification; this finishes the argument.

In general, we can reduce to the affinoid case by the following lemma.

Lemma 7.14. *Let X be a spectral topological space, and $U \subset X$ a quasicompact open subset. Then $U^{\mathrm{wl}} \to X^{\mathrm{wl}}$ is a quasicompact open embedding. If $X = \cup_{i \in I} U_i$ is a covering of X by quasicompact open subsets $U_i \subset X$, then the induced map $\sqcup U_i^{\mathrm{wl}} \to X^{\mathrm{wl}}$ is an open covering by quasicompact open subsets.*

Proof. In general X^{wl} comes with functorial maps to X (by adjunction) and X^{cons}, which is X endowed with the constructible topology, via the identification $\pi_0(X^{\mathrm{wl}}) = X^{\mathrm{cons}}$ from [BS15, Lemma 2.1.10]. In particular, there is a natural map

$$U^{\mathrm{wl}} \to X^{\mathrm{wl}} \times_{X^{\mathrm{cons}}} U^{\mathrm{cons}} \to X^{\mathrm{wl}} .$$

Here, $U^{\mathrm{cons}} \subset X^{\mathrm{cons}}$ is a quasicompact open subset; thus, it suffices to see that the map $U^{\mathrm{wl}} \to X^{\mathrm{wl}} \times_{X^{\mathrm{cons}}} U^{\mathrm{cons}}$ is a homeomorphism. By Lemma 2.5, it is enough to show that it is bijective and generalizing. Recall that $\pi_0(X^{\mathrm{wl}}) = X$, and for $x \in X$, the corresponding connected component of X^{wl} is the set $X_x \subset X$ of all generalizations of x, i.e. the localization of X at x. If $x \in U$, it follows that $U_x = X_x$; this shows bijectivity. Moreover, it shows that the map is generalizing, as desired.

The final statement follows from $X^{\mathrm{cons}} = \cup_{i \in I} U_i^{\mathrm{cons}}$. □

□

Moreover, we have the following variant.

Definition 7.15. *A perfectoid space X is strictly totally disconnected if it is qcqs and every étale cover splits.*

Repeating the construction $X \mapsto X_\infty$ countably often (here, we use our assumption that countable unions of cardinals less than κ are less than κ), we can find an affinoid pro-étale $\widetilde{X} \to X$ such that all étale covers of $\widetilde{X}$ split. Indeed, any étale cover can be refined by an affinoid étale cover. Such a cover comes from some finite level by Proposition 6.4 (iv), and becomes split at the next. □

In the following, we will analyze perfectoid spaces which are pro-étale over a strictly totally disconnected space.

Lemma 7.19. *Let X be a strictly totally disconnected perfectoid space. Let $f : Y \to X$ be a quasicompact separated map of perfectoid spaces. Then f is pro-étale if and only if for all rank-1-points $x = \mathrm{Spa}(C, \mathcal{O}_C) \in X$, the fibre $Y_x = Y \times_X x$ is isomorphic to $x \times \underline{S_x}$ for some profinite set S_x. In this case, Y is strictly totally disconnected, and $Y \to X$ is affinoid pro-étale.*

Note that the lemma applies in particular if Y is affinoid, as any map of affinoid perfectoid spaces is separated.

Proof. The condition is clearly necessary. Thus assume that $f : Y \to X$ has the property that for all rank-1-points $x = \mathrm{Spa}(C, \mathcal{O}_C) \in X$, the fibre $Y_x = Y \times_X x$ is isomorphic to $x \times \underline{S_x}$ for some profinite set S_x. We want to show that Y can be written as an inverse limit $Y = \varprojlim_i Y_i \to X$ of étale maps $Y_i \to X$, where all Y_i are affinoid.

First, note that $f : Y \to X$ factors as $Y \to X \times_{\pi_0(X)} \pi_0(Y) \to X$, where $X \times_{\pi_0(X)} \pi_0(Y) \to X$ is affinoid pro-étale, and $X \times_{\pi_0(X)} \pi_0(Y)$ is strictly totally disconnected. Thus, replacing X by $X \times_{\pi_0(X)} \pi_0(Y)$, we can reduce to the case that $\pi_0(f) : \pi_0(Y) \to \pi_0(X)$ is a homeomorphism.

We claim that in this situation, f is an injection. To check this, we can check on connected components, so we may assume that $X = \mathrm{Spa}(C, C^+)$, and then by assumption also Y is connected. To check that f is injective, it is by Proposition 5.3 enough to check injectivity on (K, K^+)-valued points, and then as f is separated, it is enough to check on $(K, \mathcal{O}_K)$-valued points. Thus, assume two rank-1-points $y_1, y_2 \in Y$ map to the same point $x \in X$. Now x is the unique rank-1-point of X, and the set of preimages of x forms a profinite set $Y_x \subset Y$ by assumption. By our assumption, Y_x contains at least two points, so we can find a closed and open decomposition $Y_x = U_{1,x} \sqcup U_{2,x}$, for some quasicompact open subsets $U_1, U_2 \subset Y$. Let $V_1, V_2 \subset Y$ be the closures of $U_{1,x}$ and $U_{2,x}$. As $U_{1,x}$ and $U_{2,x}$ are pro-constructible subsets, their closures are precisely the subsets of specializations of points in $U_{1,x}$ resp. $U_{2,x}$. As any point of Y generalizes to a unique point of Y_x, $Y = V_1 \sqcup V_2$. As V_1 and V_2 are both closed, this gives a contradiction to our assumption that Y is connected, finishing the proof that f is an injection.

Now, since $|f| : |Y| \to |X|$ is injective, the image of $|f|$ is a pro-constructible generalizing subset of X, and thus by Lemma 7.6 an intersection of subsets of the form $\{|g| \leq 1\}$ for varying $g \in H^0(X, \mathcal{O}_X)$. As such, the image is affinoid pro-étale in X, and itself (strictly) totally disconnected. Thus, replacing X by the image of f, we can assume that $|f|$ is a bijection. But then f is an isomorphism by Lemma 5.4. □

Lemma 5.4 allows us to give a purely topological description of perfectoid spaces which are pro-étale over a strictly totally disconnected space.

Definition 7.20. *Let T be a spectral space such that every connected component is a totally ordered chain of specializations. A spectral map $S \to T$ of spectral spaces is called affinoid pro-étale if the induced map $S \to T \times_{\pi_0(T)} \pi_0(S)$ is a pro-constructible and generalizing embedding. A spectral map*

Proposition 7.16. *Let X be a qcqs perfectoid space. Then X is strictly totally disconnected if and only if every connected component is of the form* $\mathrm{Spa}(C, C^+)$, *where C is algebraically closed.*

Proof. Assume first that X is strictly totally disconnected. This condition passes to closed subsets, so we can assume that X is connected. In that case, as X is in particular totally disconnected, $X = \mathrm{Spa}(K, K^+)$ for some perfectoid field K with an open and bounded valuation subring $K^+ \subset K$. If K is not algebraically closed, then for any finite extension L of K with integral closure $L^+ \subset L$ of K^+, the map $\mathrm{Spa}(L, L^+) \to \mathrm{Spa}(K, K^+)$ is a nonsplit finite étale cover. Thus, necessarily K is algebraically closed.

Conversely, assume that all connected components of X are of the form $\mathrm{Spa}(C, C^+)$. By Lemma 7.3, X is totally disconnected. To see that it is strictly totally disconnected, it is enough to see that any étale cover is locally split (as then the étale cover admits a refinement by an open cover, which is split as X is totally disconnected). We may assume that the étale cover is qcqs. But any étale cover of any connected component $\mathrm{Spa}(C, C^+)$ splits, and then the splitting extends to a small neighborhood by Proposition 6.4. □

Again, one could consider the following strengthening.

Definition 7.17. *A w-strictly local perfectoid space is a w-local perfectoid space X such that for all $x \in X$, the completed residue field $K(x)$ is algebraically closed.*

In particular, a w-strictly local space is strictly totally disconnected.

For any X, one can find strictly totally disconnected spaces over X, as the following lemma shows. The construction of this lemma is slightly different than the w-localization functor, and in particular also ensures that $\widetilde{X} \to X$ is universally open, which is not true for $X^{\mathrm{wl}} \to X$.

Lemma 7.18. *Let X be an affinoid perfectoid space. Then there is an affinoid perfectoid space $\widetilde{X}$ with an affinoid pro-étale surjective and universally open map $\widetilde{X} \to X$, such that $\widetilde{X}$ is strictly totally disconnected.*

As usual, per Convention 4.2, X is assumed to be κ-small, and we require $\widetilde{X}$ (and the implicit pro-system) to be κ-small as well.

Proof. We want to take the cofiltered limit of "all" affinoid étale covers over X. For this, fix a set of representatives $\{X_i \to X\}_{i \in I}$ of all affinoid étale and surjective spaces over X. We note that this is of cardinality less than κ: The cardinality of $\mathcal{O}_{X_i}(X_i)$ is bounded independently of i, say by λ (which is essentially the cardinality of $\mathcal{O}_X(X)$). The set of all affinoid perfectoid spaces Y for which $\mathcal{O}_Y(Y)$ has cardinality bounded by λ is also of cardinality less than κ (by thinking about all possible ways of endowing a set of cardinality $\leq \lambda$ with the structure of a perfectoid algebra, and possible rings of integral elements). For each such Y, the set of maps from Y to X has cardinality less than κ. In total, one sees that there less than κ such étale maps $Y \to X$. For each finite subset $J \subset I$, let X_J be the product of all X_i, $i \in J$, over X, which is still affinoid. Then $X_J \to X$ is étale and surjective, and the transition maps $X_J \to X_{J'}$ for $J' \subset J$ are also étale and surjective. In particular, if we set $X_\infty = \varprojlim_J X_J$, then $X_\infty \to X$ is affinoid pro-étale, and universally open. Indeed, it is open as any quasicompact open subset $U_\infty \subset X_\infty$ comes as the preimage of some quasicompact open $U_J \subset X_J$, and then U_J is also equal to the image of U_∞ in X_J, as $X_\infty \to X_J$ is surjective. Thus, the image of U_∞ in X agrees with the image of U_J in X, which is open, as étale maps are open. The same argument applies after any base change, so $X_\infty \to X$ is universally open.

$S \to T$ from a locally spectral space S is called pro-étale if S is covered by spectral subsets $S' \subset S$ for which the restriction $S' \to T$ is affinoid pro-étale.

Remark 7.21. If $S \to T$ is pro-étale and S is spectral, it is not automatic that $S \to T$ is affinoid pro-étale; the analogue of the separatedness condition can fail.

Corollary 7.22. *Let X be a strictly totally disconnected perfectoid space. Then the functor sending a map $f : Y \to X$ to $|f| : |Y| \to |X|$ induces equivalences between:*

(i) *the category of (κ-small) affinoid pro-étale perfectoid spaces $Y \to X$, and the category of affinoid pro-étale $S \to |X|$ (of cardinality less than κ);*
(ii) *the category of (κ-small) pro-étale perfectoid spaces $Y \to X$, and the category of pro-étale $S \to |X|$ (of cardinality less than κ).*

Proof. Lemma 7.19 implies that if $f : Y \to X$ is (affinoid) pro-étale, then $|f| : |Y| \to |X|$ is (affinoid) pro-étale. We construct an inverse functor. For this, given a pro-étale map $g : S \to |X|$, set $\mathcal{O}_S^+ := (g^* \mathcal{O}_X^+)^\wedge_\varpi$ and $\mathcal{O}_S = \mathcal{O}_S^+[\varpi^{-1}]$, where ϖ is a pseudouniformizer on the affinoid space X. For any point of S, one gets an induced valuation on $\mathcal{O}_S$ by pullback from X. One checks using Lemma 7.6 that if $g : S \to |X|$ is affinoid pro-étale, then this defines an affinoid perfectoid space Y; Lemma 7.19 implies that the corresponding map $f : Y \to X$ is affinoid pro-étale. In general, $S \to |X|$ is locally affinoid pro-étale, so again one gets a perfectoid space Y, pro-étale over X. □

Another important property of totally disconnected spaces is an automatic flatness assertion.

Proposition 7.23. *Let $X = \mathrm{Spa}(R, R^+)$ be a totally disconnected perfectoid space, and $f : Y = \mathrm{Spa}(S, S^+) \to X$ any map from an affinoid perfectoid space Y. Then S^+/ϖ is flat over R^+/ϖ, for any pseudouniformizer $\varpi \in R$.*

Moreover, if $|f| : |Y| \to |X|$ is surjective, then the map is faithfully flat.

Proof. Let $g : |X| \to \pi_0(X)$ be the projection. First, we check that $((g \circ f)_* \mathcal{O}_Y^+)/\varpi$ is flat over $(g_* \mathcal{O}_X^+)/\varpi$, as sheaves on $\pi_0(X)$. This can be checked on stalks. But the stalk of $(g_* \mathcal{O}_X^+)/\varpi$ at $c \in \pi_0(X)$ is given by K_c^+/ϖ, where $\mathrm{Spa}(K_c, K_c^+) \subset X$ is the connected component corresponding to c. Similarly, the stalk of $((g \circ f)_* \mathcal{O}_Y^+)/\varpi$ is given by S_c^+/ϖ, where $\mathrm{Spa}(S_c, S_c^+) = Y \times_X \mathrm{Spa}(K_c, K_c^+)$. But S_c^+ is ϖ-torsionfree, which implies that it is flat over K_c^+. By base change, S_c^+/ϖ is flat over K_c^+/ϖ, as desired.

As flatness can be checked locally, it follows that S^+/ϖ is flat over R^+/ϖ. More precisely, flatness can be checked on connected components of $\mathrm{Spec}(R^+/\varpi)$, but $\pi_0(\mathrm{Spec}(R^+/\varpi)) = \pi_0(X)$, and on each connected component, we have verified the desired flatness.

To check faithful flatness in case $|f|$ is surjective, we have to see that $\mathrm{Spec}(S^+/\varpi) \to \mathrm{Spec}(R^+/\varpi)$ is surjective. This can be checked on connected components, so we can assume that $X = \mathrm{Spa}(K, K^+)$ for some perfectoid field K with an open and bounded valuation subring $K^+ \subset K$. In that case, as $|f|$ is surjective, we can find a map $\mathrm{Spa}(L, L^+) \to Y$ with L a perfectoid field with an open and bounded valuation subring $L^+ \subset L$, such that $\mathrm{Spa}(L, L^+) \to \mathrm{Spa}(K, K^+)$ is surjective. But one can identify $\mathrm{Spa}(L, L^+) = \mathrm{Spec}(L^+/\varpi)$ and $\mathrm{Spa}(K, K^+) = \mathrm{Spec}(K^+/\varpi)$, so $\mathrm{Spec}(L^+/\varpi) \to \mathrm{Spec}(S^+/\varpi) \to \mathrm{Spec}(K^+/\varpi)$ is surjective, as desired. □

8. The pro-étale and v-topology

In this section, we define two Grothendieck topologies on the category of perfectoid spaces.

Definition 8.1. *Let* Perfd *be the category of κ-small perfectoid spaces.*

(i) *The big pro-étale site is the Grothendieck topology on* Perfd *for which a collection $\{f_i : Y_i \to X\}_{i \in I}$ of morphisms is a covering if all f_i are pro-étale, and for each quasicompact open subset $U \subset X$, there exists a finite subset $J \subset I$ and quasicompact open subsets $V_i \subset Y_i$ for $i \in J$, such that $U = \bigcup_{i \in J} f_i(V_i)$.*

(ii) *Let X be a perfectoid space. The small pro-étale site of X is the Grothendieck topology on the category of perfectoid spaces $f : Y \to X$ pro-étale over X, with covers the same as in the big pro-étale site.*

(iii) *The v-site is the Grothendieck topology on* Perfd *for which a collection $\{f_i : Y_i \to X\}_{i \in I}$ of morphisms is a covering if for each quasicompact open subset $U \subset X$, there exists a finite subset $J \subset I$ and quasicompact open subsets $V_i \subset Y_i$ for $i \in J$, such that $U = \bigcup_{i \in J} f_i(V_i)$.*

We note that the slightly technical condition on quasicompact subsets is the same condition that appears in the definition of the fpqc topology for schemes. Also, there is no "small" v-site of a perfectoid space X, as this would by design include all perfectoid spaces over X.

Our goal in this section is prove that both the pro-étale and the v-site are well-behaved. We start with some general remarks about quasicompact and quasiseparated objects. Let us quickly recall their definition.

Definition 8.2. *Let T be a topos.*

(i) *An object $X \in T$ is quasicompact if for any collection of objects $X_i \in T$, $i \in I$, with maps $f_i : X_i \to X$ such that $\bigsqcup f_i : \bigsqcup X_i \to X$ is surjective, there is a finite subset $J \subset I$ such that $\bigsqcup_{i \in J} X_i \to X$ is surjective, cf. SGA 4 VI Définition 1.1.*

(ii) *An object $X \in T$ is quasiseparated if for all quasicompact objects $Y, Z \in T$, the fibre product $Y \times_X Z \in T$ is quasicompact, cf. cf. SGA 4 VI Définition 1.13.*

(iii) *A map $f : Y \to X$ is quasicompact if for all quasicompact $Z \in T$, the fibre product $Z \times_X Y$ is quasicompact, cf. SGA 4 VI Définition 1.7.*

(iv) *A map $f : Y \to X$ is quasiseparated if $\Delta_f : Y \to Y \times_X Y$ is quasicompact, cf. SGA 4 VI Définition 1.7.*

The combination of quasicompact and quasiseparated is abbreviated to "qcqs".

In the following, we assume that T is an *algebraic* topos, cf. SGA 4 VI Définition 2.3. This means that there is a generating full subcategory $C \subset T$ consisting of qcqs objects that is stable under fibre products, and moreover for $X \in C$, the map $X \to *$ is quasiseparated. This assumption will be satisfied in all our examples. Under this assumption, it suffices to check these various conditions after pullback to objects of C; also, it suffices to check after one cover by objects of C. Thus, for example $X \in T$ is quasiseparated if and only if for one cover of X by objects $X_i \in C$, all $X_i \times_X X_j$ are quasicompact, cf. SGA 4 VI Corollaire 1.17. Also, $f : Y \to X$ is quasiseparated if and only if for all $Z \in C$ with a map $Z \to X$, the object $Y \times_X Z$ is quasiseparated, cf. SGA 4 VI Corollaire 2.6. Another equivalent characterization is that for all quasiseparated $Z \in T$ with a map $Z \to X$, the object $Y \times_X Z$ is quasiseparated, cf. SGA 4 VI Corollaire 2.8.

We warn the reader that if the final object of T is not quasiseparated (as in the case for (sheaves on) Perfd), it is not equivalent to say that X is quasicompact (or quasiseparated) and to say that the map $X \to *$ is quasicompact (or quasiseparated). It is true that X quasiseparated implies $X \to *$ is quasiseparated, and that $X \to *$ being quasicompact implies that X is quasicompact.

Indeed, the former follows from our assumption that T is algebraic by SGA 4 VI Proposition 2.2, and the latter follows from SGA 4 VI Proposition 1.3, by choosing any quasicompact object $Y \in T$, and the corresponding cover $X \times Y \to X$, where $X \times Y$ is quasicompact if $X \to *$ is.

Proposition 8.3. *The topoi of sheaves on* Perfd *for either the (big) pro-étale or v-topology, and the topos of sheaves on $X_{\text{proét}}$ for a perfectoid space X, are algebraic. A basis of qcqs objects stable under fibre products is in all cases given by affinoid perfectoid spaces. Moreover, a perfectoid space X is quasicompact resp. quasiseparated in any of these settings if and only if $|X|$ is quasicompact resp. quasiseparated.*

Proof. Left to the reader. □

Convention 8.4. Further below, we will sometimes ask that a map $f : Y' \to Y$ of stacks on some topos T is quasiseparated. Following the convention of [Sta, Tag 04YW], we define this to mean that $\Delta_f : Y' \to Y' \times_Y Y'$ is quasicompact *and quasiseparated*, where the latter condition is not automatic for stacks (as the diagonal need not be injective).

Proposition 8.5. *Let X be a perfectoid space, and $\nu : X_{\text{proét}} \to X_{\text{ét}}$ the natural map of sites.*

(i) *For any sheaf $\mathcal{F}$ on $X_{\text{ét}}$, the natural adjunction map $\mathcal{F} \to \nu_*\nu^*\mathcal{F}$ is an equivalence. If $\mathcal{F}$ is a sheaf of abelian groups, then $R^i\nu_*\nu^*\mathcal{F} = 0$ for all $i \geq 1$.*

(ii) *Let $Y = \varprojlim_i Y_i \to X_0 \subset X$ be a pro-étale presentation of an affinoid pro-étale map $Y \to X_0$ to an affinoid open subset $X_0 \subset X$, and let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. The natural map*

$$\varinjlim_i \mathcal{F}(Y_i) \to (\nu^*\mathcal{F})(Y)$$

is an isomorphism.

(iii) *The presheaves $\mathcal{O}$, $\mathcal{O}^+$ on $X_{\text{proét}}$ sending $Y \in X_{\text{proét}}$ to $\mathcal{O}_Y(Y)$ resp. $\mathcal{O}_Y^+(Y)$, are sheaves. If X is affinoid perfectoid, then $H^i(X_{\text{proét}}, \mathcal{O}) = 0$ for $i > 0$, and $H^i(X_{\text{proét}}, \mathcal{O}^+)$ is almost zero for $i > 0$.*

Proof. All assertions reduce to the case that X is affinoid. In this case, the sites $X_{\text{proét}}^{\text{aff}}$ and $X_{\text{proét}}$ define equivalent topoi, so we can work with $X_{\text{proét}}^{\text{aff}}$ instead. Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. It follows from Proposition 7.10 that $\nu^*\mathcal{F}$ is the sheafification of the presheaf $\nu^+\mathcal{F}$ sending a pro-étale presentation $Y = \varprojlim_i Y_i \to X$ to

$$(\nu^+\mathcal{F})(Y) = \varinjlim_i \mathcal{F}(Y_i) \ ;$$

in fact, this presheaf is the pullback on the level of presheaves. To prove (ii), one argues as in [BS15, Lemma 5.1.1]. More precisely, we want to show that $\nu^+\mathcal{F}$ is a sheaf. Thus, for any surjective map $\widetilde{Y} \to Y$ in $X_{\text{proét}}^{\text{aff}}$, we need to see that

$$(\nu^+\mathcal{F})(Y) \to \text{eq}((\nu^+\mathcal{F})(\widetilde{Y}) \rightrightarrows (\nu^+\mathcal{F})(\widetilde{Y} \times_Y \widetilde{Y}))$$

is an isomorphism. But we can write $\widetilde{Y} = \varprojlim_i \widetilde{Y}_i \to Y$ as a cofiltered limit of affinoid étale surjective $\widetilde{Y}_i \to Y$, and

$$(\nu^+\mathcal{F})(\widetilde{Y}) = \varinjlim_i (\nu^+\mathcal{F})(\widetilde{Y}_i) \ ,$$

and similarly for $(\nu^+\mathcal{F})(\widetilde{Y} \times_Y \widetilde{Y})$, which reduces us to the case that $\widetilde{Y} \to Y$ is étale. In that case, writing $Y = \varprojlim_j Y_j$ as a cofiltered limit of affinoid étale $Y_j \to X$, the map $\widetilde{Y} \to Y$ comes as the pullback of some affinoid étale $\widetilde{Y}_j \to Y_j$ for j large enough by Proposition 6.4 (iv). Then

$$\mathrm{eq}((\nu^+\mathcal{F})(\widetilde{Y}') \rightrightarrows (\nu^+\mathcal{F})(\widetilde{Y}' \times_Y \widetilde{Y}')) = \varinjlim_j \mathrm{eq}(\mathcal{F}(\widetilde{Y}_j') \rightrightarrows \mathcal{F}(\widetilde{Y}_j' \times_{Y_j} \widetilde{Y}_j')) = \varinjlim_j \mathcal{F}(Y_j) = \mathcal{F}(Y) ,$$

as desired.

Part (i) is proved similarly, following [BS15, Corollary 5.1.6]. Finally, for part (iii), choose a pseudouniformizer $\varpi \in \mathcal{O}_X^+(X)$. Then by [Sch12, Proposition 7.13], we have a sheaf of almost $\mathcal{O}_X^+(X)/\varpi$-modules on $X_{\text{ét}}^{\text{aff}}$, sending $Y \in X_{\text{ét}}^{\text{aff}}$ to $(\mathcal{O}_Y^+(Y)/\varpi)^a$. By part (ii), the pullback of this sheaf to the pro-étale site sends $Y = \varprojlim_i Y_i \to X$ to

$$(\varinjlim_i \mathcal{O}_{Y_i}^+(Y_i)/\varpi)^a = (\mathcal{O}_Y^+(Y)/\varpi)^a ,$$

so sending $Y \in X_{\text{proét}}^{\text{aff}}$ to $(\mathcal{O}_Y^+(Y)/\varpi)^a$ is a sheaf of almost $\mathcal{O}_X^+(X)/\varpi$-modules. By induction, one gets sheaves $(\mathcal{O}_Y^+(Y)/\varpi^n)^a$ for all $n \geq 1$, and then passing to the inverse limit, a sheaf $Y \mapsto \mathcal{O}_Y^+(Y)^a$; inverting ϖ gives a sheaf $\mathcal{O} : Y \mapsto \mathcal{O}_Y(Y)$. All of these sheaves have vanishing higher cohomology groups on X by part (i) and [Sch12, Proposition 7.13]. Finally, $\mathcal{O}^+ \subset \mathcal{O}$ is the subpresheaf of those functions which are everywhere of absolute ≤ 1, and so is a sheaf as $\mathcal{O}$ is. □

Corollary 8.6. *The presheaves $\mathcal{O} : X \mapsto \mathcal{O}_X(X)$ and $\mathcal{O}^+ : X \mapsto \mathcal{O}_X^+(X)$ on the big pro-étale site are sheaves. Moreover, the big pro-étale site is subcanonical, i.e. for every perfectoid space X, the functor $Y \mapsto \mathrm{Hom}(Y, X)$ is a sheaf for the big pro-étale site.*

Proof. As any cover $\{f_i : Y_i \to X\}$ in the big pro-étale site can also be regarded as a cover in the small pro-étale site of X, these questions reduce to the small pro-étale site of X. Then the first part follows from Proposition 8.5 (iii). For the second part, assume that $\{f_i : Y_i \to Y\}$ is a pro-étale cover of Y, and one has maps $g_i : Y_i \to X$ which agree on overlaps $Y_i \times_Y Y_j$. Note first that by Lemma 2.5, the maps $|Y_i| \to |X|$ glue to a continuous map $|Y| \to |X|$; in particular, the problem can be considered locally on X, so we can assume that $X = \mathrm{Spa}(R, R^+)$ is affinoid. Then we are given maps $(R, R^+) \to (\mathcal{O}(Y_i), \mathcal{O}^+(Y_i))$ which agree on overlaps; as $\mathcal{O}$ and $\mathcal{O}^+$ are sheaves for the pro-étale site, these maps glue to a map $(R, R^+) \to (\mathcal{O}(Y), \mathcal{O}^+(Y))$, which gives a map $Y \to X$, as desired. □

Next, we want to extend these results to the v-site. Here, the strategy is to combine pro-étale localization with the automatic faithful flatness results over totally disconnceted bases, Proposition 7.23.

Theorem 8.7. *The presheaves $\mathcal{O} : X \mapsto \mathcal{O}_X(X)$ and $\mathcal{O}^+ : X \mapsto \mathcal{O}_X^+(X)$ on the v-site are sheaves. Moreover, the v-site is subcanonical.*

Proof. As in Corollary 8.6, it is enough to show that $\mathcal{O}$ and $\mathcal{O}^+$ are sheaves. As $\mathcal{O}^+ \subset \mathcal{O}$ is the subpresheaf of functions which are everywhere of absolute value ≤ 1 (which can be checked after v-covers), it is enough to prove that $\mathcal{O}$ is a sheaf. As for every perfectoid space X, $\mathcal{O}(X)$ injects into $\prod_{x \in X} K(x)$, it is clear that $\mathcal{O}$ is separated. It remains to see that given a v-cover $\{f_i : Y_i \to X\}$ of a perfectoid space and functions $g_i \in \mathcal{O}(Y_i)$ which agree on overlaps $Y_i \times_X Y_j$, they glue uniquely to a function $g \in \mathcal{O}(X)$. This can be checked locally on X, so we can assume that X is affinoid.

In that case, the quasicompactness assumptions on a v-cover ensure that we can refine the given cover into a single $f : Y \to X$, where Y is affinoid. Let $\widetilde{X} \to X$ be a totally disconnected cover of X along an affinoid pro-étale map (for example the w-localization). We can also replace f by the composite $Y \times_X \widetilde{X} \to \widetilde{X} \to X$. By Corollary 8.6, we have descent for $\widetilde{X} \to X$, so it is enough to handle the map $Y \times_X \widetilde{X} \to \widetilde{X}$; in other words, we can assume that X is totally disconnected, so let $f : Y = \mathrm{Spa}(S, S^+) \to X = \mathrm{Spa}(R, R^+)$ be a map of affinoid perfectoid spaces, where X is totally disconnected.

But now S^+/ϖ is flat over R^+/ϖ by Proposition 7.23. In fact, as f is surjective, it is faithfully flat, so R^+/ϖ is the equalizer of the two maps from S^+/ϖ to $S^+/\varpi \otimes_{R^+/\varpi} S^+/\varpi$. But if $Y \times_X Y = \mathrm{Spa}(T, T^+)$, then the map $S^+/\varpi \otimes_{R^+/\varpi} S^+/\varpi \to T^+/\varpi$ is almost an isomorphism by [Sch12, proof of Proposition 6.18]. Thus, R^+/ϖ is almost the equalizer of the two maps from S^+/ϖ to T^+/ϖ; passing to the limit over the similar statement for ϖ^n and inverting ϖ shows that R is the equalizer of the two maps from S to T, as desired. □

In fact, the vanishing of cohomology also extends to the v-site.

Proposition 8.8. *Let X be an affinoid perfectoid space. Then $H^i_v(X, \mathcal{O}) = 0$ for $i > 0$, and $H^i_v(X, \mathcal{O}^+)$ is almost zero for $i > 0$.*

Proof. Assume that X is totally disconnected. First, we check vanishing of Cech cohomology, i.e. for any v-cover $f : Y = \mathrm{Spa}(S, S^+) \to X = \mathrm{Spa}(R, R^+)$, we show that the complex

$$0 \to R^+ \to S^+ \to \mathcal{O}^+(Y \times_X Y) \to \dots$$

is almost exact. As everything is ϖ-adically complete and ϖ-torsionfree, where $\varpi \in R^+$ is a pseudouniformizer, we can check this modulo ϖ. But then the complex is almost the same as

$$0 \to R^+/\varpi \to S^+/\varpi \to S^+/\varpi \otimes_{R^+/\varpi} S^+/\varpi \to \dots ,$$

which is exact by Proposition 7.23. Similarly, for general affinoid X, the Cech complex for the affinoid pro-étale cover $X^{\mathrm{wl}} \to X$ is almost exact by Proposition 8.5 (iii).

We claim that this implies that $H^i_v(X, \mathcal{O}^+)$ is almost zero for $i > 0$ for affinoid X. Indeed, choose i minimal for which $H^i_v(X, \mathcal{O}^+)$ is not almost zero for all affinoid X, and choose X and a class $\alpha \in H^i_v(X, \mathcal{O}^+)$ that is not almost zero. Let $f : \widetilde{X} \to X$ be a totally disconnected cover; if $f^*(\alpha)$ is almost zero in $H^i_v(\widetilde{X}, \mathcal{O}^+)$, a Cech-to-sheaf cohomology spectral sequence gives that $\check{H}^i(\widetilde{X}/X, \mathcal{O}^+)$ is not almost zero, which is a contradiction. Thus, we can replace X by $\widetilde{X}$ and assume that X is totally disconnected. But there is some v-cover, without loss of generality by an affinoid $f : Y \to X$, such that $f^*(\alpha) = 0$. A Cech-to-sheaf cohomology spectral sequence would then give that $\check{H}^i(Y/X, \mathcal{O}^+)$ is not almost zero, which contradicts the first paragraph. □

9. Descent

In this section, we establish some descent results for perfectoid spaces. As a piece of general notation, we define the category of descent data as follows.

Definition 9.1. *Let $\mathcal{C}$ be a site with fibre products, and let F be a prestack on $\mathcal{C}$, i.e. a functor from $\mathcal{C}^{\mathrm{op}}$ to groupoids. Let $f : Y \to X$ be a map in $\mathcal{C}$, $p_1, p_2 : Y \times_X Y \to Y$ the two projections, and $p_{12}, p_{13}, p_{23} : Y \times_X Y \times_X Y \to Y \times_X Y$ the three projections. Then*

$$F(Y/X) := \{(s, \alpha) \mid s \in F(Y), \alpha : p_1^*(s) \cong p_2^*(s) \in F(Y \times_X Y), p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha\} ,$$

which comes with a natural map $F(X) \to F(Y/X)$, sending $t \in F(X)$ to (s, α) with $s = f^*(t)$ and α the natural identification $p_1^*(s) = (p_1 \circ f)^*(t) = (p_2 \circ f)^*(t) = p_2^*(s)$.

This is the category of objects in $F(Y)$ equipped with a descent datum to X. Note that if f is a covering and F is a stack, then the natural map $F(X) \to F(Y/X)$ is an equivalence.

In this section, we will be interested in several examples of prestacks F on the pro-étale or v-site of perfectoid spaces.

Lemma 9.2. *Let F be the prestack on the category of perfectoid spaces sending X to the groupoid of perfectoid spaces over X. Let X be a perfectoid space, and $Y \to X$ a v-cover. The functor $F(X) \to F(Y/X)$ is fully faithful.*

Proof. Given two perfectoid spaces $X_1, X_2 \to X$, we have to see that we can glue morphisms over v-covers. This follows formally from the fact that the v-site is subcanonical. □

In the following, we want to prove that in some instances, the map is an equivalence.

Proposition 9.3. *Let F be the prestack on the category of affinoid perfectoid spaces, sending X to the groupoid of affinoid perfectoid spaces over X. Let X be a totally disconnected perfectoid space, and $Y \to X$ a v-cover, where Y is an affinoid perfectoid space. The functor $F(X) \to F(Y/X)$ is an equivalence of categories.*

Proof. Let $X = \mathrm{Spa}(R, R^+)$, $Y = \mathrm{Spa}(S, S^+)$, and $\widetilde{Y} = \mathrm{Spa}(\widetilde{S}, \widetilde{S}^+)$ an affinoid perfectoid space over Y with a descent datum to X. Choose a pseudouniformizer $\varpi \in R^+$ dividing p. By Proposition 7.23, S^+/ϖ is faithfully flat over R^+/ϖ; in particular, $(S^+/\varpi)^a$ is faithfully flat over $(R^+/\varpi)^a$. One can then use faithfully flat descent in the almost world, cf. [GR03, Section 3.4.1], to see that the $(S^+/\varpi)^a$-algebra $(\widetilde{S}^+/\varpi)^a$ descends to an $(R^+/\varpi)^a$-algebra, which is perfectoid in the sense of [Sch12, Definition 5.1 (iii)], and thus, by [Sch12, Theorem 5.2], is of the form $(\widetilde{R}^\circ/\varpi)^a$, where $\widetilde{R}$ is a perfectoid R-algebra. In particular, $\widetilde{R} \widehat{\otimes}_R S = \widetilde{S}$.

It remains to see that one can find the correct $\widetilde{R}^+ \subset \widetilde{R}$. Let $\widetilde{R}^+_{\mathrm{min}} \subset \widetilde{R}$ be the integral closure of $R^+ + \widetilde{R}^{\circ\circ} \subset \widetilde{R}^\circ$, and $\widetilde{X}' = \mathrm{Spa}(\widetilde{R}, \widetilde{R}^+_{\mathrm{min}})$. We get a map $\widetilde{Y} \to \widetilde{X}'$ such that the induced map $\widetilde{Y} \to \widetilde{Y}' := \widetilde{X}' \times_X Y$ is a pro-(open immersion): It is cut out by the conditions $|f| \leq 1$ for all $f \in \widetilde{S}^+$. Let $|\widetilde{X}| \subset \widetilde{X}'$ denote the image of $\widetilde{Y} \to \widetilde{X}'$; then $\widetilde{Y} \subset \widetilde{Y}'$ is precisely the preimage of $\widetilde{X} \subset \widetilde{X}'$ (as $|\widetilde{X}'|$ is the quotient of $|\widetilde{Y}'|$ by the equivalence relation which is the image of

$$|\widetilde{Y}' \times_{\widetilde{X}'} \widetilde{Y}'| \to |\widetilde{Y}'| \times |\widetilde{Y}'| \ ).$$

We need to see that $|\widetilde{X}| \subset \widetilde{X}'$ is cut out by conditions $|f| \leq 1$ for elements $f \in \widetilde{R}$. This is the content of the following lemma. □

Lemma 9.4. *Let $X = \mathrm{Spa}(R, R^+)$ be a totally disconnected affinoid perfectoid space, let $\widetilde{X} = \mathrm{Spa}(\widetilde{R}, \widetilde{R}^+)$ be an affinoid perfectoid space over X, and let $A \subset |\widetilde{X}|$ be a subset. Assume that there is a surjective map $Y = \mathrm{Spa}(S, S^+) \to X$ such that the preimage B of A in $|\widetilde{Y}|$, where $\widetilde{Y} = \widetilde{X} \times_X Y = \mathrm{Spa}(\widetilde{S}, \widetilde{S}^+)$, is an intersection of subsets of the form $|g| \leq 1$ for elements $g \in \widetilde{S}$. Then A is an intersection of subsets of the form $|f| \leq 1$ for elements $f \in \widetilde{R}$.*

Proof. We may assume that X is of characteristic p, as the subsets of the form $|f| \leq 1$ are unchanged under tilting. Note that B is pro-constructible, and thus the image A of B is also pro-constructible.

As a first reduction step, we reduce to the case that X is connected. Let $\pi : X \to \pi_0(X)$ be the projection. Let $\widetilde{x} \in \widetilde{X} \setminus A$ be any point, and let $c \in \pi_0(X)$ be the connected component $X_c = \pi^{-1}(c) \subset X$ whose preimage contains $\widetilde{x}$. If the result is true in the connected case, it holds true for the intersection A_c of A with $\widetilde{X}_c := \widetilde{X} \times_X X_c$. Thus, there is a function $f_c \in H^0(\widetilde{X}_c, \mathcal{O}_{\widetilde{X}_c})$ such that $|f_c(\widetilde{x})| > 1$, but $|f_c| \leq 1$ on A_c. Modulo functions in $H^0(\widetilde{X}_c, \mathcal{O}^+_{\widetilde{X}_c})$, f_c can be lifted to a function $f_U \in H^0(\widetilde{X}_U, \mathcal{O}_{\widetilde{X}_U})$ for some compact open neighborhood U of c in $\pi_0(X)$, where $\widetilde{X}_U \subset \widetilde{X}$ is the preimage of $U \subset \pi_0(X)$. Then f_U still satisfies $|f_U(\widetilde{x})| > 1$, and the pro-constructible subset $\{|f_U| > 1\} \cap A_U \subset \widetilde{X}_U$ does not meet the fibre over c. As $\{|f_U| > 1\} \cap A_U \subset \widetilde{X}_U$ is a pro-constructible subset, it is a spectral space, and hence a quasicompactness argument implies that after shrinking U to a smaller compact open neighborhood of c, $\{|f_U| > 1\} \cap A_U = \emptyset$, so that $A_U \subset \{|f_U| \leq 1\}$. Now, extending f_U to a function f on all of $\widetilde{X}$ by setting it to 0 on the preimage of $\pi_0(X) \setminus U$, we get a function f on $\widetilde{X}$ with $|f(\widetilde{x})| > 1$, but $|f| \leq 1$ on A. The intersection of $\{|f| \leq 1\}$ over all such functions thus gives A.

Thus, we can assume that $R = K$ is a perfectoid field, and $R^+ = K^+$ is an open and bounded valuation subring; fix a pseudouniformizer $\varpi \in K$. As a second reduction step, we reduce to the case that K is algebraically closed. Indeed, let C be a completed algebraic closure of K, and $C^+ \subset C$ the completion of the integral closure of K^+ in C. If the result is known in the algebraically closed case, we can assume that $Y = \mathrm{Spa}(C, C^+)$. Fix a point $\widetilde{x} \in \widetilde{X} \setminus A$, and a lift $\widetilde{y} \in \widetilde{Y} \setminus B$ of $\widetilde{x}$. By assumption, there is some function $g_{\widetilde{y}} \in \widetilde{S}$ with $|g_{\widetilde{y}}(\widetilde{y})| > 1$, but $|g_{\widetilde{y}}| \leq 1$ on B. Now Y is an inverse limit of $\mathrm{Spa}(L, L^+)$ over all finite extensions $L \subset C$ of K with integral closure $L^+ \subset L$ of K^+ in L. Approximating the function $g_{\widetilde{y}}$ modulo $\widetilde{S}^+$, we can assume that $g_{\widetilde{y}}$ is a function on $\widetilde{X} \times_X \mathrm{Spa}(L, L^+) =: \mathrm{Spa}(\widetilde{R}_L, \widetilde{R}^+_L)$, for some big enough finite extension L of K. In this case, $\widetilde{R}_L = \widetilde{R} \otimes_K L$ is a finite free $\widetilde{R}$-algebra; consider the characteristic polynomial $P_{g_{\widetilde{y}}}(X) = X^d + f_1 X^{d-1} + \ldots + f_d$ of $g_{\widetilde{y}}$ acting on $\widetilde{R}_L$, so all $f_i \in \widetilde{R}$. As $|g_{\widetilde{y}}| \leq 1$ on the preimage of A, it follows that $|f_i| \leq 1$ on A for all $i = 1, \ldots, d$. On the other hand, if we had $|f_i(\widetilde{x})| \leq 1$ for all $i = 1, \ldots, d$, then the equation

$$0 = P_{g_{\widetilde{y}}}(g_{\widetilde{y}}) = g^d_{\widetilde{y}} + f_1 g^{d-1}_{\widetilde{y}} + \ldots + f_d$$

would imply that $|g_{\widetilde{y}}(\widetilde{y})| \leq 1$, which is a contradiction. Thus, at least one f_i satisfies $|f_i| \leq 1$ on A while $|f_i(\widetilde{x})| > 1$. Intersecting all such f_i for varying $\widetilde{x}$ thus writes A as an intersection of sets of the form $\{|f| \leq 1\}$ for varying $f \in \widetilde{R}$.

Finally, assume that $R = C$ is a complete algebraically closed nonarchimedean field of characteristic p, and $R^+ = C^+$ is an open and bounded valuation subring. We may assume that similarly $S = L$ is a perfectoid field, and $S^+ = L^+$ is an open and bounded valuation subring. Fix a point $\widetilde{x} \in \widetilde{X} \setminus A$, with preimage $\widetilde{Y}_{\widetilde{x}} \subset \widetilde{Y}$, a pro-constructible subset. Let $x \in X$ be the image of $\widetilde{x}$. By assumption, for any $\widetilde{y} \in \widetilde{Y}_{\widetilde{x}}$, there is some function $g_{\widetilde{y}} \in \widetilde{S}$ with $|g_{\widetilde{y}}(\widetilde{y})| > 1$, but $|g_{\widetilde{y}}| \leq 1$ on B. By quasicompactness, we can find finitely many $g_1, \ldots, g_n$ such that $|g_i| \leq 1$ on B, but

$$\widetilde{Y}_{\widetilde{x}} \subset \bigcup_{i=1}^{n} \{|g_i| > 1\} .$$

Up to an element of $\widetilde{S}^+$, we can approximate $g_i \in \widetilde{S} = S \widehat{\otimes}_R \widetilde{R}$ as a sum $\varpi^{-n_i} \sum_{j=1}^{m_i} s_{ij} \otimes \widetilde{r}_{ij}$ with $s_{ij} \in S$, $\widetilde{r}_{ij} \in \widetilde{R}$. Let $Y' = \mathrm{Spa}(S', S'^+)$, with $S' = C\langle (T_{ij}^{1/p^\infty})_{i=1,\ldots,n, j=1,\ldots,m_i} \rangle$, $S'^+ =$

$C^+\langle (T_{ij}^{1/p^\infty})_{i=1,\dots,n,j=1,\dots,m_i}\rangle$, and consider the map $Y \to Y'$ given by $T_{ij} \mapsto s_{ij}$. Then there are functions $g_i' = \varpi^{-n_i} \sum_{j=1}^{m_i} T_{ij} \otimes \widetilde{r}_{ij} \in \widetilde{S}'$, where $\widetilde{Y}' = \widetilde{X} \times_X Y' = \mathrm{Spa}(\widetilde{S}', \widetilde{S}'^+)$. These functions pull back to g_i, and in particular they satisfy $|g_i'| \leq 1$ on B. Let $B' \subset |\widetilde{Y}'|$ be the preimage of A. Let $f : \widetilde{Y}' \to Y'$ be the projection. Let $W \subset Y'$ be the image of $Y \to Y'$; thus $W = \mathrm{Spa}(L', L'^+)$ is the set of generalizations of a point $y' \in Y'$ (given by the image of the closed point of Y), and W can thus be written as the intersection of all rational subsets $V \subset Y'$ containing W. Then $B' \cap f^{-1}(W) \subset \{|g_i'| \leq 1\}$, as B surjects onto $B' \cap f^{-1}(W)$. By a quasicompactness argument, this implies that there is some rational neighborhood $V \subset Y'$ of W such that $B' \cap f^{-1}(V) \subset \{|g_i'| \leq 1\}$. Similarly, the fibre of $\widetilde{Y}'_{\widetilde{x}}$ over W is contained in $\bigcup_{i=1}^n \{|g_i'| > 1\}$; said differently, the spectral space

$$M = \{\widetilde{y} \in \widetilde{Y}'_{\widetilde{x}} \mid \forall i = 1, \dots, n : |g_i'(\widetilde{y})| \leq 1\}$$

has empty fiber over $W \subset Y'$. By a quasicompactness argument, this implies that replacing V by a smaller rational subset of Y' containing W, the fiber of M over V is empty. Thus, we can choose V with the properties

$$B' \cap f^{-1}(V) \subset \bigcap_{i=1}^n \{|g_i'| \leq 1\}\ , \tag{9.1}$$

$$\widetilde{Y}'_{\widetilde{x}} \cap f^{-1}(V) \subset \bigcup_{i=1}^n \{|g_i'| > 1\}\ . \tag{9.2}$$

Now V is a rational subset of

$$\mathrm{Spa}(C\langle (T_{ij}^{1/p^\infty})_{i=1,\dots,n,j=1,\dots,m_i}\rangle, C^+\langle (T_{ij}^{1/p^\infty})_{i=1,\dots,n,j=1,\dots,m_i}\rangle)$$

mapping surjectively to $\mathrm{Spa}(C, C^+)$. By Lemma 9.5 this implies that there is a (C, C^+)-valued point $z : \mathrm{Spa}(C, C^+) \to V$. Let $f_i \in \widetilde{R}$ be the evaluation of g_i' at z. Then $|f_i| \leq 1$ on A by (9.1), but $\widetilde{x} \in \bigcup_{i=1}^n \{|f_i| > 1\}$ by (9.2). This finishes the proof, as now some $f = f_i$ satisfies $|f| \leq 1$ on A, but $|f(\widetilde{x})| > 1$, and intersecting $\{|f| \leq 1\}$ over all such f for varying $\widetilde{x}$ gives A. □

Lemma 9.5. *Let C be an algebraically closed nonarchimedean field with open and bounded valuation subring $C^+ \subset C$. Let V be an open subset of*

$$\mathrm{Spa}(C\langle T_1, \dots, T_d\rangle, C^+\langle T_1, \dots, T_d\rangle)$$

mapping surjectively to $\mathrm{Spa}(C, C^+)$. *Then there is a section* $\mathrm{Spa}(C, C^+) \to V$.

Proof. We may assume that V is rational, and in particular affinoid. Then $\mathfrak{X} = \mathrm{Spf}\, \mathcal{O}^+(V)$ is a formal scheme which is flat and topologically of finite type (thus, of finite presentation) over $\mathrm{Spf}\, C^+$; moreover, $\mathfrak{X} \to \mathrm{Spf}\, C^+$ is surjective. We need to see that $\mathfrak{X} \to \mathrm{Spf}\, C^+$ has a section. First, $R = C^+/C^{\circ\circ}$ is a valuation ring with algebraically closed fraction field K, and $X = \mathfrak{X} \times_{\mathrm{Spf}\, C^+} \mathrm{Spec}\, R$ is a scheme of finite presentation and faithfully flat over R. This implies that $X(R)$ is nonempty: Any finitely presented faithfully flat cover of a strictly henselian local ring is refined by a finite flat locally free cover, cf. [Gro68, after Lemma 11.2], and finite flat covers of $\mathrm{Spec}\, R$ split. But $\mathfrak{X}(C^+) = \mathfrak{X}(\mathcal{O}_C) \times_{X(K)} X(R)$, and the reduction map $\mathfrak{X}(\mathcal{O}_C) \to X(K)$ is surjective. □

We need another result for morphisms which are not necessarily affinoid. We will however need to make a separatedness assumption.

Proposition 9.6. *Let F be the prestack on the category of perfectoid spaces sending a perfectoid space X to the groupoid of separated pro-étale perfectoid spaces over X. Let X be a strictly totally disconnected perfectoid space, and $Y \to X$ a v-cover. Then $F(X) \to F(Y/X)$ is an equivalence of categories.*

Proof. By Lemma 9.2, we only have to prove that all descent data are effective; for this, we may assume that Y is strictly totally disconnected as well.

Let $R = \widetilde{Y} \times_X Y \subset \widetilde{Y} \times \widetilde{Y}$ be the equivalence relation on $\widetilde{Y}$ corresponding to the descent datum. We claim that the image $R' \subset |\widetilde{Y}| \times |\widetilde{Y}|$ satisfies the hypothesis of Lemma 2.7. First, $|\widetilde{Y}|$ is a quasiseparated locally spectral space. As all maps of perfectoid spaces are generalizing, it follows that $R' \to \widetilde{Y}$ is generalizing; also, it is quasicompact (as it is a quotient of the map $|R| = |\widetilde{Y} \times_X Y| \to |\widetilde{Y}|$, and $Y \to X$ is quasicompact). Finally, R' is pro-constructible: For any quasicompact open subset $W \subset \widetilde{Y}$, the intersection of R' with $|W| \times |W|$ is the image of the spectral space $W \times_X Y \cap Y \times_X W$ under the spectral map to $|W| \times |W|$.

Thus, by Lemma 2.7, for any quasicompact open $W \subset \widetilde{Y}$ we can find R-invariant subsets $U \subset E \subset \widetilde{Y}$ such that U is open in $\widetilde{Y}$ and contains W, and E is an intersection of a nonempty family of quasicompact open subsets. Thus, $E \subset \widetilde{Y}$ corresponds by Lemma 7.19 to an affinoid pro-étale perfectoid space, which we will still denote by E. As E is R-invariant, it comes with an induced descent datum, which is effective by Proposition 9.3. We find some affinoid perfectoid space $E_X \to X$ whose pullback to Y is E; it follows (from Lemma 7.19) that $E_X \to X$ is affinoid pro-étale. Now the open R-invariant subset $U \subset E$ descends to some open subset $U_X \subset E_X$, which is thus pro-étale over X. For varying W, the U will cover $\widetilde{Y}$, and thus the U_X will glue to give the desired space $\widetilde{X} \to X$ pro-étale over X. □

Finally, we need some stronger descent results for étale morphisms.

Proposition 9.7. *The prestack on the category of perfectoid spaces sending any perfectoid space X to the groupoid of separated étale perfectoid spaces over X is a stack in the v-topology.*

Similarly, the prestack on the category of perfectoid spaces sending any perfectoid space X to the groupoid of finite étale perfectoid spaces over X is a stack in the v-topology.

Proof. We give the proof in the more difficult case of separated étale maps; it is easy to see that the property of being finite étale passes through all the arguments.

Let F be the prestack on the category of perfectoid spaces sending any perfectoid space X to the groupoid of separated étale perfectoid spaces over X. Let $f : Y \to X$ be any v-cover; we need to prove that $F(X) \to F(Y/X)$ is an equivalence. From Lemma 9.2, we know that the functor is fully faithful, so we need to show effectivity of descent. Assume first that X is strictly totally disconnected; we may also assume that Y is strictly totally disconnected. Then Proposition 9.6 implies that any separated étale $\widetilde{Y} \to Y$ with descent data descends to a separated pro-étale $\widetilde{X} \to X$. It remains to see that $\widetilde{X} \to X$ is étale. For this, we may assume that $\widetilde{X}$ is affinoid. Under the equivalence of Corollary 7.22, we have to see that $|\widetilde{X}| \to |X|$ is a local isomorphism. But $|\widetilde{X} \times_X Y| \to |\widetilde{X}| \times_{|X|} |Y|$ is a homeomorphism, and so we know that $|\widetilde{X}| \to |X|$ becomes a local isomorphism after pullback along the surjection $|Y| \to |X|$. By Lemma 9.8 below, we see that $|\widetilde{X}| \to |X|$ is a local isomorphism, as desired.

This finishes the case that X is strictly totally disconnected. In general, we can assume that X is affinoid, and by Lemma 7.18 find an affinoid pro-étale surjective and universally open map $Y' \to X$

with Y' strictly totally disconnected. Then we have a v-cover $Y \times_X Y' \to X$ refining $Y \to X$, and we already know effectivity of descent for $Y \times_X Y' \to Y'$. In other words, we may replace Y by Y', and assume that $Y \to X$ is affinoid pro-étale and universally open. Write $Y = \varprojlim_i Y_i \to X$ as a cofiltered inverse limit of étale maps $Y_i \to X$, where all Y_i are affinoid.

Now let $\widetilde{Y} \to Y$ be a separated étale map equipped with a descent datum to X, which induces an equivalence relation $\widetilde{R} \subset \widetilde{Y} \times \widetilde{Y}$. The map $\widetilde{R} \to \widetilde{Y}$ can be identified with the projection $\widetilde{Y} \times_X Y \to \widetilde{Y}$, which is open (as $Y \to X$ is universally open). From Lemma 2.10, we see that $|\widetilde{Y}|/|\widetilde{R}|$ is a quasiseparated locally spectral space. Covering it by quasicompact open subsets, we can reduce to the case that $\widetilde{Y}$ is quasicompact.

In this situation, Proposition 6.4 (iii) says that

$$Y_{\text{ét,qc,sep}} = 2\text{-}\varinjlim_i (Y_i)_{\text{ét,qc,sep}} ,$$

so any quasicompact separated étale Y-space $\widetilde{Y}$ comes via pullback from some quasicompact separated étale Y_i-space $\widetilde{Y}_i$ for i large enough. Similarly, any descent datum is defined at some finite level, so we can replace Y by Y_i for i sufficiently large, and reduce to the case that $Y \to X$ is étale. By the variant [KL15, Proposition 8.2.20] of an argument of de Jong and van der Put, [dJvdP96, Proposition 3.2.2], one further reduces to the case that $Y \to X$ is finite étale. We can also assume that it is Galois, with Galois group G.

Finally, take any point $x \in X$. As $(X_x)_{\text{ét,qc,sep}} = 2\text{-}\varinjlim_{U \ni x} U_{\text{ét,qc,sep}}$ (and similarly for pullbacks to Y and $Y \times_X Y$), it is enough to show effectivity of descent after base change to X_x; thus we may assume that $X = \mathrm{Spa}(K, K^+)$ for some perfectoid field K with open and bounded valuation subring $K^+ \subset K$. It remains to see that in this case, if $\widetilde{Y} \in X_{\text{ét,qc,sep}}$ has a free G-action for some finite group G, then the quotient $\widetilde{Y}/G$ exists (with $\widetilde{Y} \times_{\widetilde{Y}/G} \widetilde{Y} = \widetilde{Y} \times G$). For this, we use Lemma 9.9 below to reduce to the case that $\widetilde{Y} \to X$ is finite étale, which reduces us to the algebraic case of finite étale K-algebras. □

The following two lemmas were used in the proof.

Lemma 9.8. *Let $f : S' \to S$ be a map of spectral spaces, and let $p : \widetilde{S} \to S$ be a surjective, spectral and generalizing map of spectral spaces, with pullback $\widetilde{f} : \widetilde{S}' = S' \times_S \widetilde{S} \to \widetilde{S}$. If $\widetilde{f}$ is a local isomorphism, then f is a local isomorphism.*

Proof. Let $p' : \widetilde{S}' \to S'$ denote the projection. Let $s' \in S'$ be any point, with image $s \in S$. We claim that there is a quasicompact open neighborhood $U' \subset S'$ of s' such that $f|_{U'} : U' \to S$ is injective. For this, note that $\Delta_{\widetilde{f}} : \widetilde{S}' \to \widetilde{S}' \times_{\widetilde{S}} \widetilde{S}'$ is an open immersion (as $\widetilde{f}$ is a local isomorphism); this implies by Lemma 2.5 that also $\Delta_f : S' \to S' \times_S S'$ is an open immersion. In particular, there is a quasicompact open neighborhood U' of s' in S' such that $U' \times_S U' \subset \Delta_f(S')$, i.e. $U' \to S$ is injective.

Replacing S' by U', we may assume that f is injective. In that case $\widetilde{f}$ is injective and a local isomorphism. This implies that $\widetilde{f}$ is an open immersion (as the image of $\widetilde{f}$ is open, and the map to the image is a continuous open bijective map). By Lemma 2.5, it follows that f is an open immersion. □

Lemma 9.9. *Let $X = \mathrm{Spa}(K, K^+)$, where K is a perfectoid field, and $K^+ \subset K$ is an open and bounded integrally closed subring (not necessarily a valuation subring). Let $Y \to X$ be a*

quasicompact separated étale map. Then there is a functorial factorization $Y \hookrightarrow \bar{Y} \to X$, where $Y \hookrightarrow \bar{Y}$ is a quasicompact open immersion, and $\bar{Y} \to X$ is finite étale.

Remark 9.10. This compactification is a special case of the canonical compactifications introduced in Proposition 18.6 below.

Proof. Let $X^\circ = \mathrm{Spa}(K, \mathcal{O}_K)$, and $Y^\circ = Y \times_X X^\circ$. Then Y° is finite étale over X°; in particular, $|Y^\circ|$ is a finite set of maximal points. Each of them is a pro-constructible subset, and their closures (which are given by their sets of specializations) do not intersect, as any point generalizes to a unique maximal point. Thus, replacing Y by an open and closed subset, we may assume that $Y^\circ = \mathrm{Spa}(L, \mathcal{O}_L)$ is a point, where L is a finite extension of K. Then $Y^\circ \subset Y$ is the unique maximal point, so Y is connected, and $\mathcal{O}_Y$ is the constant sheaf L. Let $\bar{Y} = \mathrm{Spa}(L, L^+)$, where L^+ is the integral closure of K^+ in L. Then $\bar{Y} \to X$ is finite étale, and there is a natural map $Y \to \bar{Y}$, given by the map $(L, L^+) \to (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ of algebras. Then $Y \to \bar{Y}$ is a separated étale map of perfectoid spaces, which after pullback along $X^\circ \subset X$ is an isomorphism. In particular, it is an isomorphism on rank-1-valued points, so by the valuative criterion for separatedness (Proposition 5.11), it is an injection (cf. Proposition 5.3 (ii)). Thus, Y is determined by the image of the injective map $|Y| \to |\bar{Y}|$, which is a quasicompact open subspace of $\bar{Y}$, as $Y \to \bar{Y}$ is étale. Therefore, $Y \to \bar{Y}$ is a quasicompact open immersion, as desired. □

Corollary 9.11. *Let $f : Y \to X$ be a map of perfectoid spaces, and let $\widetilde{X} \to X$ be a v-cover, with pullback $\widetilde{f} : \widetilde{Y} = Y \times_X \widetilde{X} \to \widetilde{X}$.*

(i) *Assume that $\widetilde{f}$ is pro-étale, and X is strictly totally disconnected. Then f is pro-étale.*

(ii) *Assume that $\widetilde{f}$ is étale. Then f is étale.*

(iii) *Assume that $\widetilde{f}$ is finite étale. Then f is finite étale.*

Proof. We can assume that X is affinoid. In cases (i) and (ii), the conclusion is local on Y, so we can in these cases assume that Y is affinoid; in particular, f is separated. Now (i) follows from Proposition 9.6, and parts (ii) and (iii) from Proposition 9.7. □

10. Morphisms of v-stacks

The descent results of the previous section make it possible to define good notions of étale and (quasi-)pro-étale morphisms between pro-étale stacks on Perfd. In the following, we will often confuse a perfectoid space and the sheaf that it represents.

Definition 10.1. *Let $f : Y' \to Y$ be a map of pro-étale stacks on the category of (κ-small) perfectoid spaces* Perfd.

(i) *Assume that f is locally separated, i.e. there is an open cover of Y' over which f becomes separated. The map f is quasi-pro-étale if for any strictly totally disconnected X with a map $X \to Y$ (i.e., a section in $Y(X)$), the pullback $Y' \times_Y X$ is representable and $Y' \times_Y X \to X$ is pro-étale.*

(ii) *Assume that f is locally separated. The map f is étale if for any perfectoid space X with a map $X \to Y$, the pullback $Y' \times_Y X$ is representable, and $Y' \times_Y X \to X$ is étale.*

(iii) *The map f is finite étale if for any perfectoid space X with a map $X \to Y$, the pullback $Y' \times_Y X \to X$ is representable, and $Y' \times_Y X \to X$ is finite étale.*

In parts (i) and (ii), we restrict the definition to the case that f is locally separated, as otherwise one may want to give a slightly different definition. Note that f is always locally separated if Y' is a perfectoid space, and Y is a v-sheaf. We make the following important convention.

Convention 10.2. In the remainder of this text, we take "locally separated" as part of the definition of étale or quasi-pro-étale maps of pro-étale stacks.

Any pro-étale morphism of perfectoid spaces is quasi-pro-étale, but the converse need not hold. This is similar to the difference between pro-étale and weakly étale morphisms of schemes, cf. [BS15].

The following proposition is immediate from the definitions.

Proposition 10.3. *Let $f : X' \to X$ be a map of perfectoid spaces. Then f is étale, resp. finite étale, if and only if the corresponding map of pro-étale sheaves on* Perfd *is étale, resp. finite étale. If X is strictly totally disconnected, then f is pro-étale if and only if the corresponding map of pro-étale sheaves on* Perfd *is quasi-pro-étale.* □

Proposition 10.4. *Let $f : Y_1 \to Y_2$, $g : Y_2 \to Y_3$ be morphisms of pro-étale stacks on* Perfd, *with composite $h = g \circ f : Y_1 \to Y_3$.*

(i) *If f and g are quasi-pro-étale (resp. étale, finite étale), then so is h.*

(ii) *If g and h are quasi-pro-étale (resp. étale, finite étale), then so is f.*

(iii) *Any pullback of a quasi-pro-étale (resp. étale, finite étale) map is quasi-pro-étale (resp. étale, finite étale).*

In some situations, one can also deduce that g is quasi-pro-étale from knowing that f and h are quasi-pro-étale, cf. Proposition 11.30 below.

Proof. These are all straightforward. Let us check the most subtle part, which is (i) in the quasi-pro-étale case. We may assume that $X_3 = Y_3$ is a strictly totally disconnected perfectoid space. In this case, Y_2 is representable and pro-étale over X_3. As such, Y_2 admits an open cover by strictly totally disconnected perfectoid spaces $X_2 \subset Y_2$. The preimage of any such $X_2 \subset Y_2$ in Y_1 is representable and pro-étale over X_2. It follows that Y_1 has an open cover by representable functors, so Y_1 is itself representable. Also $Y_1 \to Y_2$ is pro-étale locally on Y_2, thus pro-étale, and then $Y_1 \to Y_3$ is pro-étale as a composite of pro-étale maps. □

We will also need to extend other notions like injections, open and closed immersions, and separated maps to morphisms of sheaves. These notions work better for v-sheaves, because of the following result.

Proposition 10.5. *Let X be a totally disconnected perfectoid space, and let $Y \subset X$ be a sub-v-sheaf. Then Y is ind-representable. More precisely, one can write Y as a (κ-small) filtered colimit of $Y_i \subset Y \subset X$ which are pro-constructible and generalizing subsets of X, in particular affinoid pro-étale over X.*

Proof. Let Z be any affinoid perfectoid space with a map $Z \to Y$. Then the image of the composition $Z \to Y \to X$ on topological spaces is a pro-constructible and generalizing subset $\bar{Z} \subset X$. By Lemma 7.6, the subset $\bar{Z}$ is naturally an affinoid perfectoid space, and $Z \to \bar{Z}$ is a v-cover. As Y is sub-v-sheaf of X, it follows that one gets a map $\bar{Z} \to Y$. Note that the category of pro-constructible generalizing subsets $Y_i \subset X$ for which the inclusion $Y_i \to X$ factors over Y is filtered: This follows from the sheaf property of Y. This implies that Y is the filtered colimit of such Y_i, as desired. □

Corollary 10.6. *Let $f : Y' \to Y$ be a quasicompact injection of v-stacks. Then f is quasi-pro-étale, and for every totally disconnected perfectoid space X mapping to Y, the fibre product $Y' \times_Y X$ is representable by a pro-constructible and generalizing subset of X.*

Proof. We may assume that $Y = X$ is a totally disconnected perfectoid space. The previous proposition shows that $Y' \subset X$ is ind-representable. By quasicompactness of Y', this implies that Y' is representable; applying the proposition again, we see that $Y' \subset X$ is affinoid pro-étale, as desired. □

Definition 10.7. *Let $f : Y' \to Y$ be a map of pro-étale stacks.*

(i) *The map f is an open immersion if for every perfectoid space X mapping to Y, the pullback $Y' \times_Y X \to X$ is representable by an open immersion.*

(ii) *The map f is a closed immersion if for every totally disconnected perfectoid space X mapping to Y, the pullback $Y' \times_Y X \subset X$ is representable by a closed immersion.*

(iii) *The map f is separated if $\Delta_f : Y' \to Y' \times_Y Y'$ is a closed immersion.*

(iv) *The map f is 0-truncated if for all $X \in \mathrm{Perfd}$, the map of groupoids $f(X) : Y'(X) \to Y(X)$ is faithful.*

Moreover, the pro-étale stack Y is separated if the map $Y \to *$ is separated.

Remark 10.8. We will only occasionally use the last definition, with which one has to be a bit careful. Namely, it can happen that Y is separated, but Y is not quasiseparated: The issue is that the notions of Y being quasiseparated and of $Y \to *$ being quasiseparated do not coincide. (An example is given by $Y = X/\varphi^{\mathbb{Z}}$, where X is some perfectoid space of characteristic p, and φ is its absolute Frobenius: This defines a diamond Y that is separated, but not quasiseparated.)

In part (iv), it is equivalent to ask that for all affinoid perfectoid spaces (or more generally, all pro-étale sheaves) X mapping to Y, the pro-étale stack $Y' \times_Y X$ is discrete, i.e. a pro-étale sheaf. It is also equivalent to ask that $\Delta_f : Y' \to Y' \times_Y Y'$ is an injection; in particular, separated maps are automatically 0-truncated. (This differs from the convention in [Sta. Tag 04YW].)

For a map of perfectoid spaces $f : X' \to X$, these notions are equivalent to the previously defined notions. Moreover, there is again a valuative criterion for separatedness.

Proposition 10.9. *Let $f : Y' \to Y$ be a map of v-stacks. Then f is separated if and only if f is 0-truncated, quasiseparated, and for every perfectoid field K with ring of integers $\mathcal{O}_K$ and an open and bounded valuation subring K^+, and any diagram*

$$\begin{array}{ccc} \mathrm{Spa}(K, \mathcal{O}_K) & \longrightarrow & Y' \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spa}(K, K^+) & \longrightarrow & Y \end{array},$$

there exists at most one dotted arrow making the diagram commute.

Note that if f is 0-truncated, the groupoid of dotted arrows is automatically discrete, i.e. there are no automorphisms.

Proof. If f is separated, then Δ_f is a closed immersion, which implies that Δ_f is quasicompact, so that f is quasiseparated. Also, closed immersions are specializing, which shows that there is at

most one such dotted arrow. Conversely, if f is quasiseparated, then Δ_f is quasicompact. Thus, by Corollary 10.6, the diagonal Δ_f is quasi-pro-étale. It remains to check that for any totally disconnected perfectoid space X mapping to $Y' \times_Y Y'$, the pro-constructible generalizing subset

$$Y' \times_{Y' \times_Y Y'} X \subset X$$

is closed. For this, it is enough to check that it is closed under specializations, which is exactly the valuative criterion. □

Sometimes it is useful to know that a version of the valuative criterion holds true for general pairs (R, R^+).

Proposition 10.10. *Let $f : Y' \to Y$ be a separated map of v-stacks, and let R be a perfectoid Tate ring with an open and integrally closed subring $R^+ \subset R^\circ$. Then, for any diagram*

$$\begin{array}{ccc} \mathrm{Spa}(R, R^\circ) & \longrightarrow & Y' \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spa}(R, R^+) & \longrightarrow & Y \end{array},$$

there exists at most one dotted arrow making the diagram commute.

Proof. Assume that $a, b : \mathrm{Spa}(R, R^+) \to Y'$ are two maps projecting to the same map $\bar{a} = \bar{b} : \mathrm{Spa}(R, R^+) \to Y$, and with same restriction $a^\circ = b^\circ : \mathrm{Spa}(R, R^\circ) \to Y'$. Consider the sub-v-sheaf

$$Z = \mathrm{Spa}(R, R^+) \times_{Y' \times_Y Y'} Y' \subset \mathrm{Spa}(R, R^+),$$

where the two projections $\mathrm{Spa}(R, R^+) \to Y'$ are given by a and b. As f is separated, Δ_f is a closed immersion, so $Z \subset \mathrm{Spa}(R, R^+)$ is a closed immersion. To check that it is an equality, we can assume that R is totally disconnected; then Z is a totally disconnected perfectoid space as well. Now for any perfectoid field (K, K^+), we know that $Z(K, K^+) = \mathrm{Spa}(R, R^+)(K, K^+)$ by the valuative criterion, Proposition 10.9. Thus, $|Z| = |\mathrm{Spa}(R, R^+)|$, and then $Z = \mathrm{Spa}(R, R^+)$. □

The next proposition shows that many of these conditions can be checked v-locally on the target.

Proposition 10.11. *Let $f : Y' \to Y$ be a map of v-stacks, and let $g : \widetilde{Y} \to Y$ be a surjective map of v-stacks. Let $\widetilde{f} : \widetilde{Y}' = \widetilde{Y} \times_Y Y' \to \widetilde{Y}$ be the pullback of f.*

(o) *If $\widetilde{f}$ is quasicompact (resp. quasiseparated), then f is quasicompact (resp. quasiseparated).*

(i) *If $\widetilde{f}$ is an open (resp. closed) immersion, then f is an open (resp. closed) immersion.*

(ii) *If $\widetilde{f}$ is separated, then f is separated.*

(iii) *If $\widetilde{f}$ is finite étale, then f is finite étale.*

(iv) *If $\widetilde{f}$ is separated and étale, then f is separated and étale.*

(v) *If $\widetilde{f}$ is separated and quasi-pro-étale, then f is separated and quasi-pro-étale.*

Proof. For (o), the quasiseparated case reduces to the quasicompact case by passing to the diagonal. One can also replace Y by an affinoid perfectoid space X, and $\widetilde{Y}$ by a v-cover $\widetilde{X} \to X$, where $\widetilde{X}$ may be assumed to be an affinoid perfectoid space. Now let Y'_i, $i \in I$, be a set of v-sheaves with a surjection $\bigsqcup_{i \in I} Y'_i \to Y'$. The pullback $\bigsqcup_{i \in I} \widetilde{Y}'_i \to \widetilde{Y}'$ has a finite subcover by quasicompactness

of $\widetilde{Y}'$, so assume $J \subset I$ is a finite subset such that $\bigsqcup_{i\in J} \widetilde{Y}_i' \to \widetilde{Y}'$ is surjective. In particular, the composite $\bigsqcup_{i\in J} \widetilde{Y}_i' \to \widetilde{Y}' \to Y'$ is surjective, which implies that $\bigsqcup_{i\in J} Y_i' \to Y'$ is surjective.

Part (ii) reduces to (i) by passing to the diagonal. Now for (i), (iii), (iv) and (v), we need to check that something holds after any pullback $X \to Y$ for an affinoid perfectoid space X. As $\widetilde{Y} \to Y$ is surjective, we can find a v-cover $\widetilde{X} \to X$ such that $X \to Y$ lifts to $\widetilde{X} \to \widetilde{Y}$. We may then replace Y by X and $\widetilde{Y}$ by $\widetilde{X}$.

Now (i) for open immersions follows because $|\widetilde{X}| \to |X|$ is a quotient map by Lemma 2.5. For closed immersions, one uses in addition that if X is totally disconnected, then f is representable by Corollary 10.6, as $\widetilde{f}$ is a quasicompact injection, and thus f is a quasicompact injection.

Finally, parts (iii) and (iv) follow from Proposition 9.7, and part (v) (in which case one assumes that X is strictly totally disconnected) from Proposition 9.6. □

As an application, let us discuss $\underline{G}$-torsors for locally profinite groups G. Here, for any topological space T, we denote by $\underline{T}$ the associated v-sheaf given by

$$\underline{T}(X) = C^0(|X|, T) .$$

Definition 10.12. *Let G be a locally profinite group. A $\underline{G}$-torsor is map $f : \widetilde{X} \to X$ of v-stacks with an action of $\underline{G}$ on $\widetilde{X}$ over X, such that v-locally on X, there is a $\underline{G}$-equivariant isomorphism $\widetilde{X} \cong \underline{G} \times X$.*

Lemma 10.13. *Let G be a locally profinite group, X a perfectoid space, and $f : \widetilde{X} \to X$ a $\underline{G}$-torsor. Then the v-sheaf $\widetilde{X}$ is representable by a perfectoid space, and $\widetilde{X} \to X$ is pro-étale, universally open and a v-cover. More precisely, for any open subgroup $K \subset G$, pushout along the discrete G-set G/K defines a separated étale map*

$$\widetilde{X}_K := \widetilde{X} \times_{\underline{G}} \underline{G/K} \to X ,$$

the transition map $\widetilde{X}_{K'} \to \widetilde{X}_K$ is finite étale if $K' \subset K$ is of finite index, and

$$\widetilde{X} = \varprojlim_K \widetilde{X}_K \to X .$$

Related results had been obtained by Hansen, [Han16].

Proof. In case $\widetilde{X} = \underline{G} \times X$, the results are clear, as then $\widetilde{X}_K = \underline{G/K} \times X$ is a disjoint union of copies of X. Thus, Proposition 10.11 (iv) implies that $\widetilde{X}_K \to X$ is representable by a separated étale map in general, and Proposition 10.11 (iii) implies that $\widetilde{X}_{K'} \to \widetilde{X}_K$ is finite étale in case $K' \subset K$ is of finite index. Also, $\widetilde{X} = \varprojlim_K \widetilde{X}_K$, as this can be checked v-locally. This presentation shows that $\widetilde{X} \to X$ is pro-étale, as desired. It is also a v-cover, as this can be checked v-locally, where it admits a section. Finally, the map $\widetilde{X} \to X$ is universally open, as any open subspace comes via pullback from $\widetilde{X}_K$ for sufficiently small K, the map $\widetilde{X} \to \widetilde{X}_K$ is surjective, and $\widetilde{X}_K \to X$ is open (as it is étale). □

11. Diamonds

From now on, we change the setup, and work with the full subcategory $\mathrm{Perf} \subset \mathrm{Perfd}$ of (κ-small) perfectoid spaces of characteristic p.

Definition 11.1. *A diamond is a sheaf Y for the pro-étale topology of* Perf *such that Y can be written as a quotient X/R, where X is representable by a perfectoid space, and $R \subset X \times X$ is a representable equivalence relation for which the projections $s, t : R \to X$ are pro-étale.*

For convenience, we introduce the following name for the equivalence relations.

Definition 11.2. *Let X be a perfectoid space. A pro-étale equivalence relation on X is a representable equivalence relation $R \subset X \times X$ such that the projections $s, t : R \to X$ are pro-étale.*

Note that we do not make any assumptions like representability of the diagonal; the issue is that there is no good notion of morphisms that are relatively representable (in perfectoid spaces).

Proposition 11.3. *Let $X \in$ Perf, and $R \subset X \times X$ a pro-étale equivalence relation.*

(i) *The quotient sheaf $Y = X/R$ is a diamond.*
(ii) *The natural map $R \to X \times_Y X$ of sheaves on* Perf *is an isomorphism.*
(iii) *Let $\widetilde{X} \to X$ be a pro-étale cover by a perfectoid space $\widetilde{X}$, and*

$$\widetilde{R} = R \times_{X \times X} (\widetilde{X} \times \widetilde{X}) \subset \widetilde{X} \times \widetilde{X}$$

the induced equivalence relation. Then $\widetilde{R}$ is a pro-étale equivalence relation on $\widetilde{X}$, and the natural map $\widetilde{Y} := \widetilde{X}/\widetilde{R} \to Y = X/R$ is an isomorphism.
(iv) *The map $X \to Y$ is quasi-pro-étale.*

Recall that quasi-pro-étale morphisms are defined in Definition 10.1 (i).

Proof. Part (i) is clear by definition. For part (ii), note that both are subsheaves of $X \times X$, so $R \to X \times_Y X$ is injective. On the other hand, let Z be any perfectoid space with a map $Z \to X \times_Y X$. By definition, this means that we have two maps $a, b : Z \to X$ such that the two induced maps $Z \to Y$ agree. This condition means that after some pro-étale cover $\widetilde{Z} \to Z$, the composite map $\widetilde{Z} \to Z \to X \times X$ factors over the equivalence relation R. We get a map $\widetilde{Z} \to R$, which descends to a map $Z \to R$ as the two maps $\widetilde{Z} \times_Z \widetilde{Z} \to R$ coming from projection on either factor agree: Namely, their composites with $R \hookrightarrow X \times X$ agree. We have produced a map $Z \to R$, which factors the map $Z \to X \times X$, as this holds true after composition with the pro-étale cover $\widetilde{Z} \to Z$. Therefore, $R \to X \times_Y X$ is surjective, as desired.

In part (iii), first note that $\widetilde{R}$ is representable as a fibre product of representable objects, and the two projections $\widetilde{s}, \widetilde{t} : \widetilde{R} \to \widetilde{X}$ are pro-étale: For example, one can write $\widetilde{s}$ as the composite of

$$(R \times_X \widetilde{X}) \times_X (\widetilde{X} \to X) : \widetilde{R} = R \times_{X \times X} (\widetilde{X} \times \widetilde{X}) \to R \times_{X \times X} (\widetilde{X} \times X) = R \times_X \widetilde{X}$$

and

$$s \times_X \widetilde{X} : R \times_X \widetilde{X} \to \widetilde{X} ,$$

both of which are a base change of a pro-étale map, and thus pro-étale.

Now the map $\widetilde{Y} \to Y$ of pro-étale sheaves on Perf is surjective, as the composite $\widetilde{X} \to \widetilde{Y} \to Y$ is, as this can also be written as the composite $\widetilde{X} \to X \to Y$. To check whether $\widetilde{Y} \to Y$ is injective, let Z be a perfectoid space with two maps $a, b : Z \to \widetilde{Y}$ which agree after composing with $\widetilde{Y} \to Y$. Replacing Z by a pro-étale cover, we may assume that a, b factor over $\widetilde{a}, \widetilde{b} : Z \to \widetilde{X}$. The associated map $Z \to \widetilde{X} \times \widetilde{X} \to X \times X$ factors over R by assumption and part (ii), so we get a map $Z \to R \times_{X \times X} (\widetilde{X} \times \widetilde{X}) = \widetilde{R}$. This means that $\widetilde{a}, \widetilde{b} : Z \to \widetilde{X}$ induce the same map $Z \to \widetilde{Y}$, proving injectivity of $\widetilde{Y} \to Y$, thereby finishing the proof of part (iii).

For part (iv), we can replace X by $\widetilde{X} = \bigsqcup_i U_i \to X$, where the U_i form an affinoid cover of X and R by the induced equivalence relation $\widetilde{R} \subset \widetilde{X} \times \widetilde{X}$: Indeed, these spaces still satisfy the same assumptions by part (iii), and to check whether $X' \times_Y X$ is representable and pro-étale over X', it is enough to check for the open subspaces $X' \times_Y U_i$. In particular, after this replacement, $s, t : R \to X$ are separated: Indeed, both maps $R \subset X \times X$ and $X \times X \to X$ are separated.

As by definition $X \to Y$ is surjective in the pro-étale topology, we can find a pro-étale cover $\widetilde{X'} \to X'$ and a map $\widetilde{X'} \to X$ lying over $X' \to Y$. We can assume that $\widetilde{X'}$ is affinoid. Let $W = X' \times_Y X \to X'$ be the fibre product. Then

$$\widetilde{X'} \times_{X'} W = \widetilde{X'} \times_X (X \times_Y X) = \widetilde{X'} \times_X R\,,$$

which is representable and pro-étale over $\widetilde{X'}$. Moreover, as $s, t : R \to X$ are separated, also the base change $\widetilde{X'} \times_{X'} W = \widetilde{X'} \times_X R \to \widetilde{X'}$ is separated. Now applying Proposition 9.6, we see that $W \to X'$ is representable, pro-étale and separated over X'. □

Another basic fact is the following.

Proposition 11.4. *Fibre products exist in the category of diamonds.*

Proof. Let $Y_1 \to Y_3 \leftarrow Y_2$ be a diagram of diamonds. Choose presentations $Y_i = X_i/R_i$ as in the definition of a diamond, for $i = 1, 2, 3$. After replacing X_i for $i = 1, 2$ by a pro-étale cover, we can assume that there are maps $X_i \to X_3$ lying over $Y_i \to Y_3$. We can then further replace X_i by $X_i \times_{Y_3} X_3 = X_i \times_{X_3} R_3$ to assume that the maps $X_i \to Y_1 \times_{Y_3} X_3$ are surjective in the pro-étale topology.

In this case, the map $X_4 := X_1 \times_{X_3} X_2 \to Y_1 \times_{Y_3} Y_2 =: Y_4$ is surjective in the pro-étale topology. The induced equivalence relation $R_4 = X_4 \times_{Y_4} X_4$ can be calculated as $R_4 = R_1 \times_{R_3} R_2$, which is representable. It remains to see that $R_4 \to X_4$ is pro-étale. But $R_1 \to X_1$ and $R_2 \to X_2$ are pro-étale, so $R_1 \times R_2 \to X_1 \times X_2$ and its base change $R_1 \times_{X_3} R_2 \to X_1 \times_{X_3} X_2 = X_4$ are pro-étale. It remains to see that $R_4 = R_1 \times_{R_3} R_2 \to R_1 \times_{X_3} R_2$ is pro-étale; this is a base change of $R_3 \to R_3 \times_{X_3} R_3$. But the diagonal of any representable map is pro-étale by Remark 7.9. □

Another characterization of diamonds is given as follows.

Proposition 11.5. *Let Y be a pro-étale sheaf on* Perf*. Then Y is a diamond if and only if there is a surjective quasi-pro-étale morphism $X \to Y$ from a perfectoid space X. If X is a disjoint union of strictly totally disconnected spaces, then $R = X \times_Y X \subset X \times X$ is a pro-étale equivalence relation with $Y = X/R$.*

Proof. We may assume that X is a disjoint union of strictly totally disconnected spaces. In this case, by definition of quasi-pro-étale morphisms (Definition 10.1 (i)), $R = X \times_Y X \to X$ is pro-étale. As in the proof of Proposition 11.3 (iii), one finds that $Y = X/R$, so Y is a diamond. □

One can use this to obtain some stability properties for diamonds.

Proposition 11.6. *Let Y be a pro-étale sheaf on* Perf*, and assume that there is a surjective quasi-pro-étale map $Y' \to Y$, where Y' is a diamond. Then Y is a diamond.*

Proof. Choose a surjective quasi-pro-étale map $X \to Y'$. Then $X \to Y' \to Y$ is a surjective quasi-pro-étale map, so Y is a diamond by Proposition 11.5. □

Proposition 11.7. *Let $f : Y' \to Y$ be a quasi-pro-étale map of pro-étale sheaves on* Perf*, and assume that Y is a diamond. Then Y' is a diamond.*

Proof. Choose a surjective quasi-pro-étale map $X \to Y$ where X is a disjoint union of strictly totally disconnected spaces. Then $X' = Y' \times_Y X$ is representable (and pro-étale over X), and $X' \to Y'$ is a surjective quasi-pro-étale map. Now Proposition 11.5 implies that Y' is a diamond. □

Proposition 11.8. *Let X be a diamond, and let $R \subset X \times X$ be an equivalence relation such that $s, t : R \to X$ are quasi-pro-étale. Then the quotient $Y = X/R$ is a diamond.*

In the statement, R is just assumed to be a pro-étale sheaf, but note that as $s : R \to X$ is quasi-pro-étale, it follows from Proposition 11.7 that R is automatically a diamond.

Proof. We may assume that X is a separated perfectoid space. In that case, the map $X \to Y$ is a surjective separated map which is quasi-pro-étale: Indeed, its base change $R = X \times_Y X \to X$ is quasi-pro-étale, and $X \to Y$ is a v-cover, so we can apply Proposition 10.11 (v). Thus, the result follows from Proposition 11.6. □

A nice property of diamonds is that they are always sheaves for the v-topology. This resembles a result of Gabber, [Sta, Tag 0APL], that all algebraic spaces are sheaves for the fpqc topology, and our proof will follow his arguments.

Proposition 11.9. *Let Y be a diamond. Then Y is a sheaf for the v-topology.*

Proof. Choose a presentation $Y = X/R$ of Y as a quotient of a perfectoid space X by a pro-étale equivalence relation $R \subset X \times X$. By Proposition 11.3 (iii), we may assume that X is a disjoint union of totally disconnected (in particular, affinoid) perfectoid spaces.

Let Z be a perfectoid space with a v-cover $\widetilde{Z} \to Z$. First, we prove injectivity of $Y(Z) \to Y(\widetilde{Z})$. Thus, assume that $a, b : Z \to Y$ are two maps such that the composite maps $\widetilde{a}, \widetilde{b} : \widetilde{Z} \to Z \to Y$ agree. After replacing Z by a pro-étale cover (and using that Y is a pro-étale sheaf), we can assume that a, b lift to maps $a_X, b_X : Z \to X$; then the composite maps $\widetilde{a}_X, \widetilde{b}_X : \widetilde{Z} \to Z \to X$ have the property that $(\widetilde{a}_X, \widetilde{b}_X) : \widetilde{Z} \to X \times X$ factors over $R \subset X \times X$. In other words, the map $(a_X, b_X) : Z \to X \times X$ factors over R after precomposition with $\widetilde{Z} \to Z$. As R is a v-sheaf, this implies that $(a_X, b_X) : Z \to X \times X$ factors over R, showing that $a = b$, as desired.

It remains to prove surjectivity of

$$Y(Z) \to \mathrm{eq}(Y(\widetilde{Z}) \rightrightarrows Y(\widetilde{Z} \times_Z \widetilde{Z})) .$$

By standard reductions (using that F is a pro-étale sheaf and separated for the v-topology), we may assume that Z and $\widetilde{Z}$ are strictly totally disconnected. Let $\widetilde{a} : \widetilde{Z} \to Y$ be a map such that the two induced maps $\widetilde{Z} \times_Z \widetilde{Z} \to Y$ agree. By Proposition 11.3 (iv), the fibre product $\widetilde{Z} \times_Y X \to \widetilde{Z}$ is representable and pro-étale; moreover, by our assumption on X, it is also separated over $\widetilde{Z}$. Note that $\widetilde{W} = \widetilde{Z} \times_Y X \to \widetilde{Z}$ comes with a descent datum relative to $\widetilde{Z}/Z$: Indeed, the fibre product

$$\widetilde{W} \times_Z \widetilde{Z} = (\widetilde{Z} \times_Z \widetilde{Z}) \times_Y X$$

is independent of which of the two induced maps $\widetilde{Z} \times_Z \widetilde{Z} \to Y$ is chosen, as they agree by assumption. By Proposition 9.6, it follows that $\widetilde{W} \to \widetilde{Z}$ descends to a surjective separated pro-étale map $W \to Z$. In particular, $\widetilde{W} = W \times_Z \widetilde{Z} \to W$ is a v-cover, and the map $\widetilde{W} \to X$ descends to $W \to X$ by construction of the descent datum, and using that X is a v-sheaf. We have constructed a pro-étale

cover $W \to Z$ with a map $W \to X$. The induced map $W \times_Z W \to X \times X$ factors over $R \subset X \times X$, as this is true for the v-cover

$$(W \times_Z W) \times_Z \widetilde{Z} = \widetilde{W} \times_{\widetilde{Z}} \widetilde{W} = \widetilde{Z} \times_Y R$$

of $W \times_Z W$. Thus, the map composite map $W \to X \to Y$ factors over a map $Z \to Y$, which pulls back to the given map $\widetilde{Z} \to Y$. □

Another interesting stability property is the following.

Proposition 11.10. *Let $f : Y' \to Y$ be an injection of v-sheaves, where Y is a diamond. Then Y' is a diamond.*

Proof. We have to find a surjective quasi-pro-étale morphism $X' \to Y'$. Pulling back via a surjective quasi-pro-étale morphism $X \to Y$, we may assume that Y is a disjoint union of totally disconnected spaces; we may replace Y by a totally disconnected space X. In that case, Proposition 10.5 says that Y' is a filtered colimit of pro-constructible generalizing subsets $X_i \subset X$. Let $X' = \bigsqcup_i X_i$, with its natural morphism to Y'. Then $X' \to Y'$ is surjective, and quasi-pro-étale (as each map $X_i \to Y'$ is quasi-pro-étale). □

A convenient property of diamonds is that it is easy to check whether a map is an isomorphism. For an even more general statement, see Lemma 12.5 below.

Lemma 11.11. *Let $f : Y \to X$ be a qcqs map of diamonds. Then f is an isomorphism if and only if for every algebraically closed perfectoid field K with open and bounded valuation subring $K^+ \subset K$, the map $f(K, K^+) : Y(K, K^+) \to X(K, K^+)$ is a bijection.*

Proof. Pulling back by some presentation of X, we may replace X by a perfectoid space, which we can then assume to be affinoid. In that case, Y is a qcqs diamond. Write $Y = \widetilde{Y}/R$ as the quotient of a qcqs perfectoid space $\widetilde{Y}$ by a pro-étale equivalence relation R, which is still qcqs. The map $\widetilde{Y} \to X$ is surjective in the v-topology, as it is a map of qcqs perfectoid spaces which by assumption is surjective on points, i.e. a v-cover. As both Y and X are v-sheaves by Proposition 11.9, it suffices to see that the map $R \to \widetilde{Y} \times_X \widetilde{Y}$ is an isomorphism. Now both R and $\widetilde{Y} \times_X \widetilde{Y}$ are qcqs perfectoid spaces. Moreover, the map $R(K, K^+) \to (\widetilde{Y} \times_X \widetilde{Y})(K, K^+)$ is a bijection for all (K, K^+) as in the statement of the lemma. Thus, the result follows from Lemma 5.4. □

Let us give a slightly weird example of diamonds.

Example 11.12. Fix a perfectoid field K of characteristic p. We claim that there is a fully faithful functor from the category of (κ-small) compact Hausdorff spaces to the category of diamonds over $\mathrm{Spa}(K, \mathcal{O}_K)$. Indeed, for any compact Hausdorff space T, one has the functor $\underline{T}$ on Perf which takes a perfectoid space X to $C^0(|X|, T)$, where C^0 denotes the continuous functions. We claim that the functor

$$T \mapsto \underline{T} \times \mathrm{Spa}(K, \mathcal{O}_K)$$

is a fully faithful functor from the category of compact Hausdorff spaces to the category of diamonds over $\mathrm{Spa}(K, \mathcal{O}_K)$.

Let us check first that it takes values in diamonds. If S is a profinite set, then we can write S as an inverse limit of finite sets S_i, and if we also denote by S_i the corresponding constant sheaf on

Perf, then $\underline{S} = \varprojlim S_i$, and $\underline{S} \times \mathrm{Spa}(K, \mathcal{O}_K) = \varprojlim (S_i \times \mathrm{Spa}(K, \mathcal{O}_K))$ is an affinoid perfectoid space, given by

$$\underline{S} \times \mathrm{Spa}(K, \mathcal{O}_K) = \mathrm{Spa}(C^0(S, K), C^0(S, \mathcal{O}_K)) .$$

Thus, profinite sets are mapped to affinoid perfectoid spaces. Now recall that any compact Hausdorff space T admits a surjection $S \to T$ from a profinite set S: One can take for S the Stone-Cech compactification of T considered as a discrete set. The induced equivalence relation $R = S \times_T S \subset S \times S$ is a closed subspace of the profinite set $S \times S$, and thus a profinite set itself. Now one can form the equivalence relation

$$\underline{R} \times \mathrm{Spa}(K, \mathcal{O}_K) \subset (\underline{S} \times \mathrm{Spa}(K, \mathcal{O}_K)) \times_{\mathrm{Spa}(K, \mathcal{O}_K)} (\underline{S} \times \mathrm{Spa}(K, \mathcal{O}_K)) ,$$

which is representable and pro-étale over $\underline{S} \times \mathrm{Spa}(K, \mathcal{O}_K)$. Let Y be the quotient $\underline{S} \times \mathrm{Spa}(K, \mathcal{O}_K)/\underline{R} \times \mathrm{Spa}(K, \mathcal{O}_K)$. We claim that $Y = \underline{T} \times \mathrm{Spa}(K, \mathcal{O}_K)$. Certainly, there is a natural map $Y \to \underline{T} \times \mathrm{Spa}(K, \mathcal{O}_K)$. It is injective: If we have two maps $a, b : Z \to \underline{S} \times \mathrm{Spa}(K, \mathcal{O}_K)$ whose composites to $\underline{T} \times \mathrm{Spa}(K, \mathcal{O}_K)$ agree, then we have two continuous maps $|Z| \to S$ whose composite to T agree, i.e. a continuous map $|Z| \to R = S \times_T S$, or in other words a map $Z \to \underline{R} \times \mathrm{Spa}(K, \mathcal{O}_K)$. For surjectivity, let Z be any perfectoid space with a map $Z \to \underline{T} \times \mathrm{Spa}(K, \mathcal{O}_K)$. We want to find a pro-étale cover $\widetilde{Z} \to Z$ and a lift $\widetilde{Z} \to \underline{S} \times \mathrm{Spa}(K, \mathcal{O}_K)$. We may assume that Z is affinoid. In that case any continuous map $|Z| \to T$ to a compact Hausdorff space T factors through the profinite set $\pi_0(Z)$ of connected components. We get an induced map $\pi_0(Z) \to T$, and a cover by the profinite set $\pi_0(Z) \times_T S \subset \pi_0(Z) \times S \to \pi_0(Z)$. It follows that

$$\widetilde{Z} = Z \times_{\pi_0(Z)} (\pi_0(Z) \times_T S)$$

is affinoid perfectoid, pro-étale over Z, and admits a lift $|\widetilde{Z}| \to S$ of $|Z| \to T$, as desired.

Finally, to check that $T \mapsto \underline{T} \times \mathrm{Spa}(K, \mathcal{O}_K)$ is fully faithful, we need to prove that the map

$$C^0(T_1, T_2) \to \mathrm{Hom}_{\mathrm{Spa}(K, \mathcal{O}_K)}(\underline{T_1} \times \mathrm{Spa}(K, \mathcal{O}_K), \underline{T_2} \times \mathrm{Spa}(K, \mathcal{O}_K))$$

is bijective for all compact Hausdorff sets T_1, T_2. For injectivity, we can assume that T_1 is a point. Now any map $\mathrm{Spa}(K, \mathcal{O}_K) \to \underline{T_2} \times \mathrm{Spa}(K, \mathcal{O}_K)$ is given by a map $|\mathrm{Spa}(K, \mathcal{O}_K)| \to T_2$ whose image recovers the given point of T_2.

For surjectivity, assume first that T_1 is profinite. In that case, $\underline{T_1} \times \mathrm{Spa}(K, \mathcal{O}_K)$ is representable, and so

$$\mathrm{Hom}_{\mathrm{Spa}(K, \mathcal{O}_K)}(\underline{T_1} \times \mathrm{Spa}(K, \mathcal{O}_K), \underline{T_2} \times \mathrm{Spa}(K, \mathcal{O}_K)) = \underline{T_2}(\underline{T_1} \times \mathrm{Spa}(K, \mathcal{O}_K)) = C^0(T_1, T_2)$$

by definition, using that $|\underline{T_1} \times \mathrm{Spa}(K, \mathcal{O}_K)| = T_1$.

Now choose a surjection $S_1 \to T_1$ from a profinite set. Given any map $\underline{T_1} \times \mathrm{Spa}(K, \mathcal{O}_K) \to \underline{T_2} \times \mathrm{Spa}(K, \mathcal{O}_K)$, we get a composite map $\underline{S_1} \times \mathrm{Spa}(K, \mathcal{O}_K) \to \underline{T_2} \times \mathrm{Spa}(K, \mathcal{O}_K)$, which comes from a map $S_1 \to T_2$ by what we have just shown. The two induced maps $S_1 \times_{T_1} S_1 \to T_2$ agree by faithfulness. Thus, we get a map from the quotient $T_1 \to T_2$ (recalling that any surjective map of compact Hausdorff spaces is a quotient map), which induces the given map $\underline{T_1} \times \mathrm{Spa}(K, \mathcal{O}_K) \to \underline{T_2} \times \mathrm{Spa}(K, \mathcal{O}_K)$ (as this is true after precomposing with the pro-étale cover $\underline{S_1} \times \mathrm{Spa}(K, \mathcal{O}_K) \to \underline{T_1} \times \mathrm{Spa}(K, \mathcal{O}_K)$).

Although compact Hausdorff spaces may have some appeal, we will generally work with spaces whose behaviour is closer to that of schemes. They are essentially characterized by having a spectral underlying topological space.

Proposition 11.13. *Let Y be a diamond, and let $Y = X/R$ be a presentation as a quotient of a perfectoid space X by a pro-étale equivalence relation $R \subset X \times X$. There is a canonical bijection between $|X|/|R|$ and the set*

$$|Y| = \{\mathrm{Spa}(K, K^+) \to Y\}/\sim ,$$

where (K, K^+) runs over all pairs of a perfectoid field K with an open and bounded valuation subring $K^+ \subset K$, and two such maps $\mathrm{Spa}(K_i, K_i^+) \to Y$, $i = 1, 2$, are equivalent, if there is a third pair (K_3, K_3^+), and a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Spa}(K_1, K_1^+) & & \\ & \nearrow & & \searrow & \\ \mathrm{Spa}(K_3, K_3^+) & & \longrightarrow & & Y \\ & \searrow & & \nearrow & \\ & & \mathrm{Spa}(K_2, K_2^+) & & \end{array} ,$$

where the maps $\mathrm{Spa}(K_3, K_3^+) \to \mathrm{Spa}(K_i, K_i^+)$, $i = 1, 2$, are surjective.

Moreover, the quotient topology on $|Y|$ induced by the surjection $|X| \to |Y|$ is independent of the choice of presentation $Y = X/R$.

Proof. First, note that the given relation $\sim$ on $\{\mathrm{Spa}(K, K^+) \to Y\}$ is an equivalence relation. This follows from the observation that for a diagram $(K_1, K_1^+) \leftarrow (K_0, K_0^+) \to (K_2, K_2^+)$ of perfectoid fields with open and bounded valuation subrings for which $\mathrm{Spa}(K_i, K_i^+) \to \mathrm{Spa}(K_0, K_0^+)$ is surjective for $i = 1, 2$, one can find another such pair (K_3, K_3^+) sitting in a diagram

$$\begin{array}{ccccc} & & \mathrm{Spa}(K_1, K_1^+) & & \\ & \nearrow & & \searrow & \\ \mathrm{Spa}(K_3, K_3^+) & & \longrightarrow & & \mathrm{Spa}(K_0, K_0^+) \\ & \searrow & & \nearrow & \\ & & \mathrm{Spa}(K_2, K_2^+) & & \end{array} ,$$

with all transition maps surjective. Indeed, this follows from surjectivity of

$$|\mathrm{Spa}(K_1, K_1^+) \times_{\mathrm{Spa}(K_0, K_0^+)} \mathrm{Spa}(K_2, K_2^+)| \to |\mathrm{Spa}(K_1, K_1^+)| \times_{|\mathrm{Spa}(K_0, K_0^+)|} |\mathrm{Spa}(K_2, K_2^+)| .$$

There is a canonical map $|X| \to |Y|$ by first interpreting $|X|$ as the similar set of equivalence classes of maps $\mathrm{Spa}(K, K^+) \to X$, which is a standard fact. This map is surjective. Indeed, given a map $\mathrm{Spa}(K, K^+) \to Y$, one can lift it to X after a pro-étale cover. Thus, after increasing K, we can lift it to a map $\mathrm{Spa}(K, K^+) \to X$, as desired. One checks directly that this map $|X| \to |Y|$ factors over $|X|/|R|$.

Now assume that two maps $\mathrm{Spa}(K_i, K_i^+) \to X$, $i = 1, 2$, project to the same point of $|Y|$. By definition of $|Y|$, this means that up to enlarging K_1 and K_2, we can assume that $K = K_1 = K_2$, $K^+ = K_1^+ = K_2^+$, and the composite maps $\mathrm{Spa}(K, K^+) \rightrightarrows X \to Y$ agree. In other words, we get a

map $\mathrm{Spa}(K, K^+) \to R = X \times_Y X$, proving that $|X|/|R| \to |Y|$ is injective, finishing the proof that $|X|/|R|$ maps bijectively to $|Y|$.

Finally, to see that the quotient topology on $|Y|$ is independent of the presentation, choose another presentation $Y = X'/R'$. Then $X \times_Y X'$ is a diamond, so we can find a pro-étale surjection $X'' \to X \times_Y X'$ from a perfectoid space X''. Now $|X''| \to |X|$ and $|X''| \to |X'|$ are quotient maps by Lemma 2.5, so the quotient topologies on $|Y|$ induced by the surjection from $|X|$, $|X''|$ and $|X'|$ are equivalent. □

In particular, the following definition makes sense.

Definition 11.14. *Let Y be a diamond, and write $Y = X/R$ as the quotient of a perfectoid space X by a pro-étale equivalence relation $R \subset X \times X$. The underlying topological space of $|Y|$ is the quotient space $|X|/|R|$.*

It is easy to see that this construction is functorial in Y. One can read off the open subspaces of a diamond Y on the topological space $|Y|$.

Proposition 11.15. *Let Y be a diamond with underlying topological space $|Y|$. Any open subfunctor of Y is a diamond. Moreover, there is bijective correspondence between open immersions $U \subset Y$ and open subsets $V \subset |Y|$, given by sending U to $V = |U|$, and V to the subfunctor $U \subset Y$ of those maps $X \to Y$ for which $|X| \to |Y|$ factors over $V \subset |Y|$.*

Moreover, if $f : Y' \to Y$ is a map of diamonds which is a surjection of v-sheaves, then $|f| : |Y'| \to |Y|$ is a quotient map.

Proof. Write $Y = X/R$ as usual. Given an open immersion $U \subset Y$, one gets an open subspace $U_X \subset X$ which is stable under the equivalence relation R; then $U = U_X/(R \cap (U_X \times U_X))$, where $R \cap (U_X \times U_X) \subset U_X \times U_X$ is a pro-étale equivalence relation, so that U is a diamond.

In particular, in this situation $|U_X| \subset |X|$ is an open $|R|$-invariant subspace of $|X|$, giving rise to the open subspace $V = |U| \subset |X|$. Conversely, if $V \subset |Y|$ is open, then the preimage $V_X \subset |X|$ is an open $|R|$-invariant subset, giving rise an open R-invariant subspace $U_X \subset X$, whose quotient $U_X/R \subset Y$ defines an open immersion. One checks easily that the two processes are inverse.

The final statement follows from the fact that the prestack sending a v-sheaf Y to its open subsheaves is a v-stack, which follows from Proposition 10.11 (i). □

Example 11.16. If Y is a perfectoid space, then $|Y|$ is the usual underlying topological space of Y. In the context of Example 11.12, if T is a compact Hausdorff space and K is a perfectoid field, then

$$|\underline{T} \times \mathrm{Spa}(K, \mathcal{O}_K)| = T\ .$$

In particular, the example shows that in general $|Y|$ can be quite far from a spectral space.

Definition 11.17. *Let Y be a diamond. Then Y is* spatial *if Y is quasicompact and quasiseparated, and $|Y|$ admits a basis of open subsets given by $|U|$ for quasicompact open immersions $U \subset Y$. More generally, Y is* locally spatial *if Y admits an open cover by spatial diamonds.*

We note that any perfectoid space is locally spatial, and it is spatial precisely when it is qcqs.

Proposition 11.18. *Let Y be a spatial diamond.*

(i) *The underlying topological space $|Y|$ is spectral.*

(ii) *Any quasicompact open subfunctor $U \subset Y$ is spatial.*

map $\mathrm{Spa}(K, K^+) \to R = X \times_Y X$, proving that $|X|/|R| \to |Y|$ is injective, finishing the proof that $|X|/|R|$ maps bijectively to $|Y|$.

Finally, to see that the quotient topology on $|Y|$ is independent of the presentation, choose another presentation $Y = X'/R'$. Then $X \times_Y X'$ is a diamond, so we can find a pro-étale surjection $X'' \to X \times_Y X'$ from a perfectoid space X''. Now $|X''| \to |X|$ and $|X''| \to |X'|$ are quotient maps by Lemma 2.5, so the quotient topologies on $|Y|$ induced by the surjection from $|X|$, $|X''|$ and $|X'|$ are equivalent. □

In particular, the following definition makes sense.

Definition 11.14. *Let Y be a diamond, and write $Y = X/R$ as the quotient of a perfectoid space X by a pro-étale equivalence relation $R \subset X \times X$. The underlying topological space of $|Y|$ is the quotient space $|X|/|R|$.*

It is easy to see that this construction is functorial in Y. One can read off the open subspaces of a diamond Y on the topological space $|Y|$.

Proposition 11.15. *Let Y be a diamond with underlying topological space $|Y|$. Any open subfunctor of Y is a diamond. Moreover, there is bijective correspondence between open immersions $U \subset Y$ and open subsets $V \subset |Y|$, given by sending U to $V = |U|$, and V to the subfunctor $U \subset Y$ of those maps $X \to Y$ for which $|X| \to |Y|$ factors over $V \subset |Y|$.*

Moreover, if $f : Y' \to Y$ is a map of diamonds which is a surjection of v-sheaves, then $|f| : |Y'| \to |Y|$ is a quotient map.

Proof. Write $Y = X/R$ as usual. Given an open immersion $U \subset Y$, one gets an open subspace $U_X \subset X$ which is stable under the equivalence relation R; then $U = U_X/(R \cap (U_X \times U_X))$, where $R \cap (U_X \times U_X) \subset U_X \times U_X$ is a pro-étale equivalence relation, so that U is a diamond.

In particular, in this situation $|U_X| \subset |X|$ is an open $|R|$-invariant subspace of $|X|$, giving rise to the open subspace $V = |U| \subset |X|$. Conversely, if $V \subset |Y|$ is open, then the preimage $V_X \subset |X|$ is an open $|R|$-invariant subset, giving rise an open R-invariant subspace $U_X \subset X$, whose quotient $U_X/R \subset Y$ defines an open immersion. One checks easily that the two processes are inverse.

The final statement follows from the fact that the prestack sending a v-sheaf Y to its open subsheaves is a v-stack, which follows from Proposition 10.11 (i). □

Example 11.16. If Y is a perfectoid space, then $|Y|$ is the usual underlying topological space of Y. In the context of Example 11.12, if T is a compact Hausdorff space and K is a perfectoid field, then

$$|\underline{T} \times \mathrm{Spa}(K, \mathcal{O}_K)| = T \ .$$

In particular, the example shows that in general $|Y|$ can be quite far from a spectral space.

Definition 11.17. *Let Y be a diamond. Then Y is* spatial *if Y is quasicompact and quasiseparated, and $|Y|$ admits a basis of open subsets given by $|U|$ for quasicompact open immersions $U \subset Y$. More generally, Y is* locally spatial *if Y admits an open cover by spatial diamonds.*

We note that any perfectoid space is locally spatial, and it is spatial precisely when it is qcqs.

Proposition 11.18. *Let Y be a spatial diamond.*

(i) *The underlying topological space $|Y|$ is spectral.*

(ii) *Any quasicompact open subfunctor $U \subset Y$ is spatial.*

(iii) *For any perfectoid space X' with a map $X' \to Y$, the induced map of locally spectral spaces $|X'| \to |Y|$ is spectral and generalizing.*

Proof. For part (i), first note that $|Y|$ is quasicompact, as by Proposition 11.15, any open cover of $|Y|$ determines an open cover of Y. Note that the set of quasicompact open immersions $U \subset Y$ is stable under finite intersections (as Y is quasiseparated); it follows that the set of $|U| \subset |Y|$ for U running over quasicompact open immersions forms a basis of quasicompact open subsets stable under finite intersections. It remains to check that $|Y|$ is sober, i.e. every irreducible closed subset has a unique generic point. Uniqueness holds because $|Y|$ is T_0 by Lemma 2.7. For existence of generic points, write $Y = X/R$, where X is a qcqs perfectoid space and R is a pro-étale equivalence relation, so that $|X| \to |Y|$ is a surjective spectral map of spectral spaces. Let $Z \subset |Y|$ be an irreducible closed subset, with preimage $W \subset |X|$. It is enough to see that the intersection of all nonempty quasicompact open subsets of Z is nonempty, as any point in the intersection is a generic point for Z. But the preimages of quasicompact open subsets of Z are quasicompact open subsets of W, and by quasicompactness of the constructible topology on W, their intersection is nonempty; thus, the intersection is nonempty in Z, as desired.

In (ii), any quasicompact open subfunctor $U \subset Y$ is given by a quasicompact open subset $|U| \subset |Y|$. As Y is quasiseparated, U is quasiseparated. Moreover, if $V \subset Y$ is a quasicompact open subfunctor, then $V \times_Y U \subset U$ is a quasicompact open subfunctor. Now the $|V \times_Y U| \subset |U|$ form a basis of open subsets, verifying that U is spatial.

For part (iii), we may assume that X' is qcqs, in which case $|X'|$ and $|Y|$ are spectral. We need to check that the preimage of a quasicompact open is quasicompact. If $V \subset |Y|$ is a quasicompact open, then it is covered by finitely many $|U_i| \subset |Y|$ for quasicompact open $U_i \subset Y$, so it remains to check that $X' \times_Y U_i \subset X'$ is quasicompact. But this follows from the assumption that Y is quasiseparated. To see that $|X'| \to |Y|$ is generalizing, write $Y = X/R$ as usual, with X strictly totally disconnected. Then we can replace X' by $X' \times_Y X$, and assume that $X' \to Y$ lifts to $X' \to X$. Now $|X'| \to |X|$ is generalizing, so it remains to see that $|X| \to |Y|$ is generalizing. But this follows from Lemma 2.9. □

Let us note that some of these properties extend to locally spatial diamonds.

Proposition 11.19. *Let Y be a locally spatial diamond.*

(i) *The underlying topological space $|Y|$ is locally spectral.*

(ii) *Any open subfunctor $U \subset Y$ is locally spatial.*

(iii) *The functor Y is quasicompact (resp. quasiseparated) if and only if $|Y|$ is quasicompact (resp. quasiseparated).*

(iv) *For any locally spatial diamond Y' with a map $Y' \to Y$, the induced map of locally spectral spaces $|Y'| \to |Y|$ is spectral and generalizing.*

Proof. Left as an exercise to the reader. □

We need some permanence properties of the class of (locally) spatial diamonds.

Proposition 11.20. *Let $f : Y' \to Y$ be a quasicompact injection of v-sheaves, where Y is a locally spatial diamond. Then Y' is a locally spatial diamond, $|Y'| \subset |Y|$ is a pro-constructible and generalizing subset (with the subspace topology), and the map*

$$Y' \to Y \times_{\underline{|Y|}} \underline{|Y'|}$$

of v-sheaves is an isomorphism.

Proof. We may assume that Y is spatial. Let $X \to Y$ be a surjective quasi-pro-étale map where X is a strictly totally disconnected perfectoid space. By Corollary 10.6, $X' = X \times_Y Y'$ is a pro-constructible and generalizing subspace of X; in particular $X' = X \times_{\underline{|X|}} \underline{|X'|}$. The spectral map $|X| \to |Y|$ of spectral spaces is generalizing, so the image $V \subset |Y|$ of $|X'|$ in $|Y|$ is a pro-constructible and generalizing subset, with $|X'| = |X| \times_{|Y|} V$. Also, if $R = X \times_Y X$ is the equivalence relation, which is affinoid pro-étale over X, then $R' = R \times_Y Y'$ is a pro-constructible and generalizing subset of R. One has a homeomorphism $|X'|/|R'| = V$: Indeed, the map $|X'|/|R'| \to V$ is a continuous bijective map, and $|X'| \to V$ is a quotient map by Lemma 2.5. Thus, $|Y'| = V$, and any quasicompact open subset of V comes via pullback from a quasicompact open subset of $|Y|$, which corresponds to a quasicompact open subspace of Y, and pulls back to a quasicompact open subspace of Y': Thus, Y' is spatial. The natural map $Y' \to Y \times_{\underline{|Y|}} \underline{|Y'|}$ becomes an isomorphism after pullback along $X \to Y$. As it is a map of v-sheaves, we see that $Y' = Y \times_{\underline{|Y|}} \underline{|Y'|}$, as desired. □

Lemma 11.21. *Let Y be a (locally) spatial diamond, and let $Y' \to Y$ be a finite étale map of pro-étale sheaves. Then Y' is a (locally) spatial diamond.*

Proof. We may assume that Y is spatial. By Proposition 11.7, Y' is a diamond. Choose a presentation $Y = X/R$ as a quotient of an affinoid perfectoid space X, and let $X' = X \times_Y Y'$, which is finite étale over X. Let R and R' be the respective equivalence relations. We need to find enough quasicompact open subspaces of Y'. Pick any point $y' \in |Y'|$ with image $y \in |Y|$. Let $Y_y \subset Y$ be the localization of Y at y, i.e. the intersection of all quasicompact open subspaces containing y. For any qcqs perfectoid space Z over Y, let $Z_y = Z \times_Y Y_y$. Then quasicompact open subspaces of Z_y come via pullback from quasicompact open subspaces of $Z \times_Y U$ for some quasicompact open subspace U of Y, and any two such extensions agree after shrinking U. By applying this to $Z = X'$ and $Z = R'$, we find that any quasicompact open subspace of Y'_y extends to a quasicompact open subspace of Y'_U for some U. This reduces us to the case $Y = Y_y$.

Thus, we may assume that $|Y|$ has a unique closed point. In this case, we can take $X = \mathrm{Spa}(C, C^+)$ for some algebraically closed field C with an open and bounded valuation subring $C^+ \subset C$. Then X' is a finite disjoint union of copies of X. This implies that any pro-constructible generalizing subset of X' is open. In particular, for any quasicompact open subset $V \subset X'$, the R'-invariant subset $R' \cdot V \subset X'$ is open (as it is pro-constructible and generalizing). This descends to a quasicompact open subspace of Y', and these give a basis, as desired. □

Lemma 11.22. *Let Y_i, $i \in I$, be a cofiltered inverse system of diamonds with qcqs transition maps, where I has cardinality less than κ. Assume that there is some strong limit cardinal $\kappa' < \kappa$ and κ'-small perfectoid spaces X_i surjecting onto Y_i. Then $Y = \varprojlim_i Y_i$ is a diamond with a bijective continuous map $|Y| \to \varprojlim_i |Y_i|$, and the maps $Y \to Y_i$ are qcqs. If all Y_i are (locally) spatial, then Y is (locally) spatial, and $|Y| \to \varprojlim_i |Y_i|$ is a homeomorphism.*

Proof. For the first part, it is enough to see that for any ordinal λ (less than κ), if I is the category of ordinals $\mu < \lambda$, then the cofiltered limit along I exists. We can assume that Y_0 (and then all Y_i) are qcqs. Given λ, we fix $\kappa_\lambda < \kappa$ a strong limit cardinal with cofinality larger than λ. We can also assume that $\kappa_\lambda \geq \kappa'$. In this situation, we will lift the diagram Y_μ, $\mu < \lambda$, to a diagram of

κ_λ-small strictly totally disconnected spaces X_μ, $\mu < \lambda$, with compatible quasi-pro-étale surjections $X_\mu \to Y_\mu$; in fact, with quasi-pro-étale surjections

$$X_\mu \to Y_\mu \times_{\varprojlim_{\mu'<\mu} Y_{\mu'}} \varprojlim_{\mu'<\mu} X_{\mu'} .$$

That this can be done follows easily by transfinite induction (noting that $\varprojlim_{\mu'<\mu} X_{\mu'}$ is still κ_λ-small, as the cofinality of κ_λ is larger than λ), and we note that one obtains a quasi-pro-étale surjection

$$\varprojlim_{\mu<\lambda} X_\mu \to \varprojlim_{\mu<\lambda} Y_\mu$$

in the limit. Indeed, after pullback to any strictly totally disconnected Z, this morphism is a transfinite composition of surjective affinoid pro-étale morphisms (using Lemma 7.19).

In this construction, one has

$$|Y| = |\varprojlim_{\mu<\lambda} X_\mu|/|\varprojlim_{\mu<\lambda} R_\mu| = \varprojlim_{\mu<\lambda} |X_\mu|/|R_\mu| = \varprojlim_{\mu<\lambda} |Y_\mu| ,$$

as sets, where $R_\mu = X_\mu \times_{Y_\mu} X_\mu$. If all Y_i are spatial, then Lemma 2.11 applied to the inverse system $|Y_\mu|$ and the map from $|\varprojlim_{\mu<\lambda} X_\mu|$ to the inverse limit shows that

$$|\varprojlim_{\mu<\lambda} X_\mu| \to \varprojlim_{\mu<\lambda} |Y_\mu|$$

is a quotient map, and thus $|Y| \to \varprojlim_{\mu<\lambda} |Y_\mu|$ is a homeomorphism. Moreover, any quasicompact open subspace of $|Y|$ comes via pullback from some quasicompact open subspace of $|Y_i|$, which corresponds to a quasicompact open subspace of Y_i, which in turn pulls back to a quasicompact open subspace of Y; thus, Y is spatial. If the Y_i are just locally spatial, one can fix a spatial open subset of some Y_{i_0}, and taking the preimage gives a spatial open subset of Y; also $|Y| = \varprojlim_i |Y_i|$ follows. □

Moreover, in this situation, an analogue of Proposition 6.4 holds true.

Proposition 11.23. *Let Y_i, $i \in I$, be a cofiltered inverse system of qcqs diamonds with set-theoretic bounds as in Lemma 11.22, with inverse limit $Y = \varprojlim_i Y_i$.*

(i) *Let $Y_{\text{fét}}$ denote the category of finite étale diamonds over Y, and similarly for Y_i. The base change functors $(Y_i)_{\text{fét}} \to Y_{\text{fét}}$ induce an equivalence of categories*

$$2\text{-}\varinjlim_i (Y_i)_{\text{fét}} \to Y_{\text{fét}} .$$

(ii) *Let $Y_{\text{ét,qcqs}}$ denote the category of (locally separated) étale qcqs diamonds over Y, and similarly for Y_i. The base change functors $(Y_i)_{\text{ét,qcqs}} \to Y_{\text{ét,qcqs}}$ induce an equivalence of categories*

$$2\text{-}\varinjlim_i (Y_i)_{\text{ét,qcqs}} \to Y_{\text{ét,qcqs}} .$$

(iii) *Let $Y_{\text{ét,qc,sep}} \subset Y_{\text{ét,qcqs}}$ be the full subcategory of quasicompact separated étale diamonds over Y (and similarly for Y_i). Then the base change functors $(Y_i)_{\text{ét,qc,sep}} \to Y_{\text{ét,qc,sep}}$ induce an equivalence of categories*

$$2\text{-}\varinjlim_i (Y_i)_{\text{ét,qc,sep}} \to Y_{\text{ét,qc,sep}} .$$

Proof. As in Lemma 11.22, we can assume that I is the set of ordinals $\mu < \lambda$ for some fixed ordinal λ. We pick strictly totally disconnected perfectoid spaces X_μ mapping compatibly to Y_μ as in the proof of Lemma 11.22. Let $R_\mu = X_\mu \times_{Y_\mu} X_\mu$ be the induced equivalence relations, and let $X = \varprojlim_\mu X_\mu$, $R = \varprojlim_\mu R_\mu$.

Let F be the prestack on the category of diamonds of finite étale, resp. quasicompact separated étale morphisms. Then F is a stack for the v-topology by Proposition 10.11. Note that $F(X/Y)$ involves the value of F on X, R, and $R \times_X R$, all of which are affinoid perfectoid spaces, and similarly for $F(X_\mu/Y_\mu)$. By Proposition 6.4, we see that

$$F(Y) = F(X/Y) = 2\text{-}\varinjlim_{\mu<\lambda} F(X_\mu/Y_\mu) = 2\text{-}\varinjlim_{\mu<\lambda} F(Y_\mu) ,$$

proving parts (i) and (iii). For part (ii), essentially the same argument applies, except that a priori the functor $F(Y) \to F(X/Y)$ is only fully faithful for the prestack F of qcqs étale morphisms. To check essential surjectivity in (ii), pick any étale qcqs map $f : \widetilde{Y} \to Y$, where it is understood that f is locally separated. In particular, one can cover $\widetilde{Y}$ by quasicompact open subsets $\widetilde{Y}_j \subset \widetilde{Y}$ such that $f|_{\widetilde{Y}_j}$ is separated. By (iii), this comes via pullback from some finite level. Moreover, the gluing data between the different $\widetilde{Y}_j$ will also be defined at some finite level, and the cocycle condition satisfied. This produces a (locally separated) étale qcqs map to some finite stage with pullback $\widetilde{Y}$, as desired. □

To prove further permanence properties of spatial diamonds, we need the following result.

Proposition 11.24. *Let Y be a spatial diamond. Then one can find a strictly totally disconnected perfectoid space X with a surjective and universally open quasi-pro-étale map $f : X \to Y$ that can be written as a cofiltered inverse limit of étale maps which are composites of quasicompact open immersions and finite étale maps.*

Conversely, assume that Y is a qcqs diamond such that Y admits a surjective and universally open quasi-pro-étale map $X \to Y$, where X is a perfectoid space. Then Y is a spatial diamond.

Remark 11.25. Of course, the proposition implies the more general converse that if Y is a qcqs diamond such that Y admits a surjective and universally open quasi-pro-étale map $Y' \to Y$, where Y' is a (locally) spatial diamond, then Y is a spatial diamond.

Proof. The converse is easy: If $X \to Y$ is surjective and universally open quasi-pro-étale, and without loss of generality X is affinoid, and in fact strictly totally disconnected (cf. Lemma 7.18), then the induced equivalence relation $R \subset X \times X$ is a qcqs perfectoid space, for which the maps $s, t : R \to X$ are open; thus, $|Y| = |X|/|R|$ is spectral and $|X| \to |Y|$ is spectral by Lemma 2.10, which implies that Y is spatial.

Now assume that Y is spatial and consider the set I of isomorphism classes of surjective étale maps $Y' \to Y$ that can be written as a composite of quasicompact open embeddings and finite étale maps. (To see that this is of cardinality less than κ, realize these by descent data along some fixed surjective map from a κ-small affinoid perfectoid space. Moreover, each Y' admits a surjection from a κ'-small affinoid perfectoid space for some fixed strong limit cardinal $\kappa' < \kappa$.) By Lemma 11.21, all such Y' are spatial diamonds. Choose a representative $Y_i \to Y$ for each $i \in I$. For any subset $J \subset I$, let Y_J be the product of all Y_i, $i \in J$, over Y, which is still a composite of quasicompact open embeddings and finite étale maps over Y, and thus a spatial diamond. By Lemma 11.22, the